Unit 2: Roots and Powers

2.1 Number Sets

Natural Numbers – positive counting numbers, not zero

1, 2, 3, 4, ...

Whole Numbers – positive counting numbers, including zero

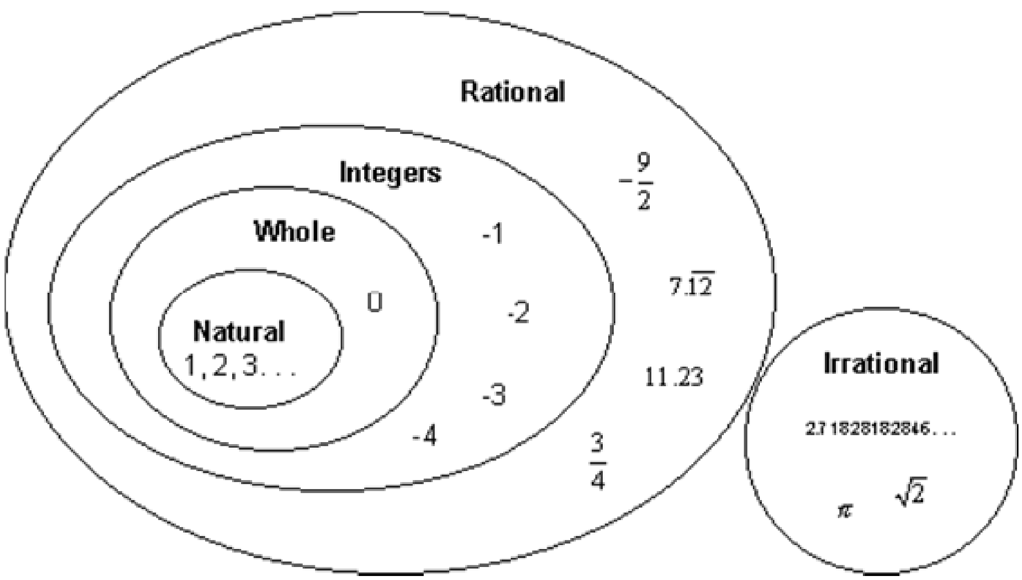
0, 1, 2, 3, 4, ...

Integers – positive and negative counting numbers

..., −2, −1, 0, 1, 2, ...

Rational Numbers – any number that can be written as a fraction $\frac{a}{b}$, $b \neq 0$, including both terminating and repeating decimals Natural, Whole, Integer

Irrational Numbers – any number that cannot be written as a fraction, a non-terminating, non-repeating decimal π



irrational

To which number system(s) does each number belong? (N, W, I, Q, $\overline{\text{Q}}$)

rational

a. 0 W, I, Q

b. $\sqrt{16}$ = 4 N, W, I, Q

c. -5 I, Q

d. $\frac{-3}{7}$ Q

e. -0.57 Q

f. $\sqrt{7}$ $\overline{Q}$

g. $\frac{1}{2}$ Q

h. -9 I, Q

i. -3 I, Q

j. 3.2 Q

k. 1 N, W, I, Q

l. 0.9999… Q

~~m. 1~~

n. $\frac{5}{2}$ Q

o. π $\overline{Q}$

Memorize these!

x	x^2	x^3	x^4
1	1	1	1
2	4	8	16
3	9	27	81
4	16	64	216
5	25	125	625
6	36	216	
7	49	343	
8	64	512	
9	81	729	
10	100	1000	10000
11	121		
12	144		
13	169		
14	196		
15	225		
16	256		
20	400		

Investigation

1. Write the two consecutive perfect squares closest to 20. Estimate the value of $\sqrt{20}$. Square your estimate. Use this value to revise your estimate. Keep revising your estimate until the square of the estimate is within 1 decimal place of 20.

$\sqrt{16}$ $\sqrt{20}$ $\sqrt{25}$

4 4.5 5

$4.5^2 = 20.25$

$4.4^2 = 19.36$

$4.47^2 = 19.98$

$\therefore \sqrt{20} \approx 4.47$

2. Write the two consecutive perfect cubes closest to 20. Estimate the value of $\sqrt[3]{20}$. Cube your estimate. Use this value to revise your estimate. Keep revising your estimate until the cube of the estimate is within 1 decimal place of 20.

$\sqrt[3]{8}$ $\sqrt[3]{20}$ $\sqrt[3]{27}$

2 2.7 3

$2.7^3 = 19.683$

$2.71^3 = 19.9$

$\therefore \sqrt[3]{20} \approx 2.71$

3. Write the two consecutive perfect fourth powers closest to 20. Use a strategy similar to that in Steps A and B to estimate a value for $\sqrt[4]{20}$.

$\sqrt[4]{16}$ $\sqrt[4]{20}$ $\sqrt[4]{81}$

2 2.1 3

$2.1^4 = 19.44$

$2.15^4 = 21.36$

$2.11^4 = 19.82$

$2.112^4 = 19.89$

$2.113^4 = 19.93$

$\sqrt[4]{20} \approx 2.113$

4. Complete this table. Use a calculator to determine the value of each radical.

Radical	Value	Is the Value Exact or Approximate?
$\sqrt{16}$	4	Exact
$\sqrt{27}$	5.1962	Approximate
$\sqrt{\frac{16}{81}}$	$\frac{4}{9}$ or $0.\overline{4}$	Exact
$\sqrt{0.64}$	0.8	Exact
$\sqrt[3]{16}$	2.5198421	Approximate
$\sqrt[3]{27}$	3	Exact
$\sqrt[3]{\frac{16}{81}}$	0.582..	Approx.
$\sqrt[3]{0.64}$	0.861..	Approx.
$\sqrt[3]{-0.64}$	−0.861..	Approx.
$\sqrt[4]{16}$	2	Exact
$\sqrt[4]{27}$	2.279...	Approx.
$\sqrt[4]{\frac{16}{81}}$	$0.\overline{6}$	Exact
$\sqrt[4]{0.64}$	0.894...	Approx.

$4 \times \sqrt{(16/81)}$

5. How can you tell if the value of a radical is a rational number?

$\sqrt{\ }$

The decimal needs to be repeating or terminating

6. How can you tell if the value of a radical is NOT a rational number?

The decimal <u>isn't</u> repeating or terminating

2.2 Rational Numbers

Example 1: Classifying Numbers

Tell whether each number is rational or irrational. Explain how you know.

a. $-\frac{3}{5}$ Rational ⇒ Fraction

b. $\sqrt{14}$ $= 3.741\ldots$

Irrational b/c the decimal doesn't end or repeat

c. $\sqrt[3]{\frac{8}{27}}$ $= 0.\overline{6}$ ⇒ Rational ⇒ The decimal repeats

$\sqrt[3]{(8/27)}$

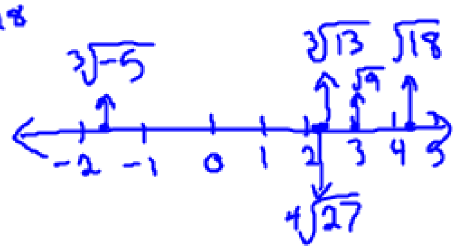
$\frac{\sqrt[3]{8}}{\sqrt[3]{27}} = \frac{2}{3}$

Example 2: Ordering Irrational Numbers on a Number Line

Use a number line to order these numbers from least to greatest.

$\sqrt[3]{13}, \sqrt{18}, \sqrt{9}, \sqrt[4]{27}, \sqrt[3]{-5}$

2.35, 4.24, 3, 2.28, -1.71



2.3 Radicals

Simplifying Radicals

**Note: $\sqrt[2]{a} \cdot \sqrt{b} = \sqrt{ab}$

$\sqrt{3} \cdot \sqrt{3} = \sqrt{3 \times 3} = \sqrt{9} = 3$

Steps for simplifying radicals: → break it down

1. Write the prime factorization of your radicand.
2. Determine the index of the radical.
3. If the index is 2, circle groups of 2 identical numbers. If the index is 3, circle groups of 3 identical number, etc.
4. The number or variable from each circled group will show up outside the radical symbol one time.
5. Anything left uncircled will remain inside the radical. If everything is circle, the radical disappears!
6. Multiply the numbers and variables outside the radical together.

Simplify:

a) $\sqrt{49} = 7$

b) $\sqrt{81} = 9$

c) $\sqrt[3]{27} = 3$

d) $\sqrt[4]{16} = 2$

e) $\sqrt{50} = \sqrt{25}\sqrt{2}$
$= 5\sqrt{2}$

f) $\sqrt[3]{16} = \sqrt[3]{8}\sqrt[3]{2}$
$= 2\sqrt[3]{2}$

g) $-2\sqrt{27} = -2\sqrt{9}\sqrt{3}$
$= -2(3)\sqrt{3}$
$= -6\sqrt{3}$

$\sqrt{3^2} = \sqrt{9} = 3$

h) $\sqrt{x^2} = x$

i) $\sqrt{x^5} = \sqrt{x^2}\sqrt{x^2}\sqrt{x} \quad = x\,x\sqrt{x}$
$= x^2\sqrt{x}$

$(x^2)(x^2)(x^1) = x^5$

m) $\sqrt[3]{144} = \sqrt[3]{8}\ \sqrt[3]{18}$
$= 2\sqrt[3]{18}$

n) $\sqrt[3]{32} = \sqrt[3]{8}\ \sqrt[3]{4}$
$= 2\sqrt[3]{4}$

o) $\sqrt[4]{32} = \sqrt[4]{16}\ \sqrt[4]{2}$
$= 2\sqrt[4]{2}$

p) $\sqrt[4]{162} = \sqrt[4]{81}\ \sqrt[4]{2}$
$= 3\sqrt[4]{2}$

q) $\sqrt{25x^2} = \sqrt{25}\sqrt{x^2}$
$= 5x$

$\sqrt[3]{x^{\boxed{3}}}$

t) $\sqrt{16x^4y^2} = \sqrt{16}\ \sqrt{x^4}\sqrt{y^2}$
$= 4\sqrt{x^2}\sqrt{x^2}\ y$
$= 4\,x\,x\,y = 4x^2y$

u) $\sqrt[3]{32x^4w^3} = \sqrt[3]{32}\ \sqrt[3]{x^4}\ \sqrt[3]{w^3}$
$= \sqrt[3]{8}\ \sqrt[3]{4}\ \sqrt[3]{x^3}\ \sqrt[3]{x}\ \sqrt[3]{w^3} = 2\sqrt[3]{4}\ x\sqrt[3]{x}\ w$
$= 2xw\sqrt[3]{4x}$

Example 2: Write as a single radical:

a) $4\sqrt{3} = \sqrt{16}\ \sqrt{3}$
$= \sqrt{16\times 3} = \sqrt{48}$

b) $3\sqrt[3]{2} = \sqrt[3]{27}\ \sqrt[3]{2}$
$= \sqrt[3]{27\times 2} = \sqrt[3]{54}$

c) $2\sqrt[5]{2} = \sqrt[5]{32}\ \sqrt[5]{2} = \sqrt[5]{32\times 2} = \sqrt[5]{64}$

$2\times2\times2\times2\times2$

2.4 Rational Exponents and Radicals

Properties of Exponents:

1) $a^m a^n = a^{m+n}$ $\quad 2^4 2^6 = 2^{10} \quad\quad 2^4 + 2^6$

2) $\dfrac{a^m}{a^n} = a^{m-n}$ $\quad \dfrac{2^6}{2^4} = 2^2$

3) $(a^n)^m = a^{nm}$ $\quad (2^2)^3 = 2^6$

4) $(ab)^m = a^m b^m$ $\quad (xy)^3 = x^3y^3$

5) $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$ $\quad \left(\dfrac{x}{y}\right)^4 = \dfrac{x^4}{y^4}$

6) $a^{-n} = \dfrac{1}{a^n}$ $\quad 4^{-2} = \dfrac{1}{4^2} \quad\quad 2^{-2} = \dfrac{1}{2^2} = \dfrac{1}{4} = 0.25$

7) $a^0 = 1$ $\quad 1.76523^0 = 1$

Complete each table below. Use a calculator to complete the second column.

x	$x^{\frac{1}{2}}$
1	$1^{\frac{1}{2}} =$ 1
4	$4^{\frac{1}{2}} =$ 2
9	$9^{\frac{1}{2}} = 3$
16	$16^{\frac{1}{2}} = 4$
25	$25^{\frac{1}{2}} = 5$

1^(1/2)

x	$x^{\frac{1}{3}}$
1	$1^{\frac{1}{3}} = 1$
8	$8^{\frac{1}{3}} = 2$
27	$27^{\frac{1}{3}} = 3$
64	$64^{\frac{1}{3}} = 4$
125	$125^{\frac{1}{3}} = 5$

$x^{\frac{1}{4}} = \sqrt[4]{x}$

Generalize the pattern in the table to determine a rule for $x^{\frac{1}{n}}$.

$$x^{\frac{1}{n}} = \sqrt[n]{x}$$

$x^{\frac{1}{2}} = \sqrt[2]{x} \quad\quad x^{\frac{1}{3}} = \sqrt[3]{x}$

Example 1: Write in radical form and evaluate (no calculator):

a. $27^{\frac{1}{3}}$ $= \sqrt[3]{27} = 3$

b. $0.49^{\frac{1}{2}}$ $= \sqrt{0.49} = 0.7$

c. $(-64)^{\frac{1}{3}}$ $= \sqrt[3]{-64} = -4$

d. $\left(\frac{4}{9}\right)^{\frac{1}{2}}$ $= \sqrt{\frac{4}{9}} = \frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$

Powers with Rational Exponents:

When m and n are natural numbers, and x is a rational number,

$$x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^{m} = \sqrt[n]{x^m} \text{ and } x^{\frac{m}{n}} = (x^m)^{\frac{1}{n}} = \sqrt[n]{x^m}$$

Example 2: Rewriting Powers in Radical and Exponent Form

a. Write the following powers as a radical.

i. $9^{\frac{3}{2}}$ $\sqrt[2]{9^3}$

ii. $\left(9^{\frac{1}{3}}\right)^{\frac{3}{2}}$ $\sqrt[3]{9^{\frac{3}{2}}}$ or $9^{\frac{1}{3}\left(\frac{3}{2}\right)} = 9^{\frac{3}{6}} = 9^{\frac{1}{2}} = \sqrt{9}$

cube root

iii. $1.8^{1.4}$ $= 1.8^{\frac{14}{10}} = \sqrt[10]{1.8^{14}}$

b. Write $\sqrt{3^5}$ and $\left(\sqrt[3]{25}\right)^2$ in exponent form.

$\sqrt[2]{3^5} = 3^{\frac{5}{2}}$

$\left(\sqrt[3]{25}\right)^2 = \left(25^{\frac{1}{3}}\right)^2 = 25^{\frac{2}{3}}$

Example 3: Evaluating Powers with Rational Exponents and Rational Bases
Evaluate.

a. $0.04^{\frac{3}{2}}$

$0.04\hat{}(3/2)$ or $\sqrt[2]{0.04^3}$

$= 0.008$ $= 0.008$

b. $27^{\frac{4}{3}}$

$27\hat{}(4/3) = 81$ or $\sqrt[3]{27^4}$

$= \sqrt[3]{531441}$

$= 81$

c. $(-32)^{0.4}$

$(-32)\hat{}0.4 = 4$

d. $1.8^{1.4}$ $= 2.277096\ldots$

Example 4: Applying Rational Exponents

Biologists use the formula $b = 0.01m^{\frac{2}{3}}$ to estimate the brain mass, b kilograms, of a mammal with body mass m kilograms. Estimate the brain mass of a husky with a body mass of 27 kg.

$27 = m$ $\rightarrow$ $b = 0.01\underbrace{(27^{\frac{2}{3}})}_{\text{type in first!}}$

$b = 0.01(9)$

$b = 0.09$

$\therefore$ The brain mass of a husky is 0.09 kg.

2.5 Negative Exponents and Reciprocals

Investigation: Complete the table below.

Quotient	Repeated Multiplication	Answer as a fraction	Use the quotient rule to find a power in the form a^m
$\frac{2^2}{2^5}$	$\frac{\cancel{2}\times\cancel{2}}{\cancel{2}\times\cancel{2}\times 2\times 2\times 2}$	$\frac{1}{2\times 2\times 2}=\frac{1}{8}$	2^{-3}
$\frac{3^4}{3^7}$	$\frac{\cancel{3}\times\cancel{3}\times\cancel{3}\times\cancel{3}}{\cancel{3}\times\cancel{3}\times\cancel{3}\times\cancel{3}\times 3\times 3\times 3}$	$\frac{1}{3\times 3\times 3}=\frac{1}{3^3}=\frac{1}{27}$	3^{-3}
$\frac{1^2}{1^5}$		$\frac{1}{1^3}=1$	1^{-3}
$\frac{4^7}{4^9}$		$\frac{1}{4^2}=\frac{1}{16}$	4^{-2}
$1=\frac{5^0}{25^4}=25^0$		$\frac{1}{25^4}$	25^{-4}
$\frac{a^4}{a^9}$		$\frac{1}{a^5}$	a^{-5}
$\frac{x^1y^5}{x^2y^6}$		$\frac{1}{xy}$	$x^{-1}y^{-1}$

What does a negative exponent mean? Look at the two columns on the right and try to create a rule.

Flip the term with the negative exponent to the denominator, make the exponent positive.

Are positive expressions with negative exponents negative?

No.

Try to simplify $\frac{1}{2^{-3}}$. $2^3=8$

Negative Exponent Law:
To change the sign of an exponent, take the reciprocal.

$$a^{-n}=\frac{1}{a^n}$$

Example 1
Simplify.

1. $8^{-1} = \frac{1}{8^1} = \frac{1}{8}$

2. $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

3. $\frac{1}{2^{-1}} = \frac{2^1}{1} = \frac{2}{1} = 2$

4. $\left(\frac{2}{3}\right)^{-1} = \frac{1}{\left(\frac{2}{3}\right)^1} = \frac{1}{\frac{2}{3}} = \frac{3}{2} \leftarrow \begin{matrix} 1 \div \frac{2}{3} \\ 1 \times \frac{3}{2} \end{matrix}$

5. $(3^{-2})(2^2) = 2^2\left(\frac{1}{3^2}\right) = \frac{2^2}{3^2} = \frac{4}{9}$

6. $x^{-4} = \frac{1}{x^4}$

7. $\left(-\frac{3}{4}\right)^{-3} = \left(-\frac{4}{3}\right)^3 = \frac{(-4)^3}{3^3} = -\frac{64}{27}$

8. $0.3^{-4} = \frac{1}{0.3^4} = \frac{1}{0.0081}$

9. $8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{\left(\sqrt[3]{8}\right)^2} = \frac{1}{2^2} = \frac{1}{4}$

10. $\left(\frac{9}{16}\right)^{-\frac{3}{2}} = \left(\frac{16}{9}\right)^{\frac{3}{2}} = \left(\frac{\sqrt{16}}{\sqrt{9}}\right)^3 = \left(\frac{4}{3}\right)^3 = \frac{4^3}{3^3} = \frac{64}{27}$

11. $(-125)^{-\frac{1}{3}} = \frac{1}{(-125)^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{-125}} = \frac{1}{-5} = -\frac{1}{5}$

2.6 Applying the Exponent Laws

Example 1: Write in simplified exponent form.

1. $\dfrac{(1.4)^3 \cdot (1.4)^4}{(1.4)^{-2}} = \dfrac{1.4^7}{1.4^{-2}} = (1.4^7)(1.4^2) = 1.4^9$

2. $\left[\left(-\dfrac{3}{2}\right)^{-4}\right]^2 \left[\left(\dfrac{3}{2}\right)^2\right]^3 = \left(-\dfrac{3}{2}\right)^{-8}\left(\dfrac{3}{2}\right)^6$

− 've
+ 've
even # of negatives makes answer +ve ← $\left(-\dfrac{2}{3}\right)^8\left(\dfrac{3}{2}\right)^6 = \left(\dfrac{2^8}{3^8}\right)\left(\dfrac{3^6}{2^6}\right) = \dfrac{2^8 3^6}{3^8 2^6} = \dfrac{2^2}{3^2} = \dfrac{4}{9}$

3. $x^{-4} = \dfrac{1}{x^4}$

4. $(x^3y^2)(x^2y^{-4}) = x^5y^{-2} = \dfrac{x^5}{y^2}$

5. $\dfrac{10x^3y^5}{2xy^3} = 5x^2y^2$

$\left(-\dfrac{4}{3}\right)6 = -\dfrac{24}{3} = -8$

6. $\left(\dfrac{7^{2/3}}{7^{1/3} \cdot 7^{5/3}}\right)^6 = \left(\dfrac{7^{2/3}}{7^{6/3}}\right)^6 = \left(7^{-4/3}\right)^6 = 7^{-8} = \dfrac{1}{7^8}$

7. $\dfrac{\left(x^{2/3}\right)\left(x^{1/3}\right)}{\left(x^{1/6}\right)} = \dfrac{x^{3/3}}{x^{1/6}} = \dfrac{x^{6/6}}{x^{1/6}} = x^{5/6}$

8. $\dfrac{\sqrt{x}\cdot\sqrt[3]{x^2}}{\sqrt[4]{x}} = \dfrac{x^{\frac{1}{2}}\,x^{\frac{2}{3}}}{x^{\frac{1}{4}}} = \dfrac{x^{\frac{6}{12}}\,x^{\frac{8}{12}}}{x^{\frac{3}{12}}} = \dfrac{x^{\frac{14}{12}}}{x^{\frac{3}{12}}} = x^{\frac{11}{12}}$

9. $\left(8x^3y^6\right)^{1/3} = \sqrt[3]{8x^3y^6} = \sqrt[3]{8}\,\sqrt[3]{x^3}\,\sqrt[3]{y^6}$

$= 2xy^2$

10. $\left(\dfrac{100a}{25a^5b^{-1/2}}\right)^{1/2} = \dfrac{100^{\frac{1}{2}}\,a^{\frac{1}{2}}}{25^{\frac{1}{2}}\,a^{\frac{5}{2}}\,b^{-\frac{1}{4}}} = \dfrac{10b^{\frac{1}{4}}}{5a^{\frac{4}{2}}}$ $\quad \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$

$= \dfrac{2b^{\frac{1}{4}}}{a^2}$

11. $\dfrac{\left(y^{1/2}\right)^3}{\left(4y^4\right)^{1/2}} = \dfrac{y^{\frac{3}{2}}}{4^{\frac{1}{2}}\,y^{\frac{4}{2}}} = \dfrac{1}{2y^{\frac{1}{2}}}$

12. $\sqrt{4}\sqrt{4} = 4^{\frac{1}{2}}\,4^{\frac{1}{2}} = 4^1$

13. $\sqrt[3]{8}\,\sqrt[3]{8} = 8^{\frac{1}{3}}\,8^{\frac{1}{3}} = 8^{\frac{2}{3}}$

14. $\sqrt[3]{64^2} = 64^{\frac{2}{3}}$

15. $\sqrt[3]{\sqrt{8}} = \left(8^{\frac{1}{2}}\right)^{\frac{1}{3}} = 8^{\frac{1}{6}}$

16. $\sqrt[5]{x^3} \cdot \sqrt[3]{x^2} =$ $x^{\frac{3}{5}} x^{\frac{2}{3}} = x^{\frac{9}{15}} x^{\frac{10}{15}} = x^{\frac{19}{15}}$

17. $\sqrt[3]{8\sqrt{x^2}} =$ $\sqrt[3]{8x} = (8x)^{\frac{1}{3}} = 8^{\frac{1}{3}} x^{\frac{1}{3}}$
$= 2x^{\frac{1}{3}}$

18. $\sqrt[3]{\sqrt{3x^5}} =$ $\left(\left(3x^5\right)^{\frac{1}{2}}\right)^{\frac{1}{3}} = \left(3x^5\right)^{\frac{1}{6}} = 3^{\frac{1}{6}} x^{\frac{5}{6}}$

19. $\sqrt{36x^5} \cdot \sqrt{49x^2} =$ $\left(36x^5\right)^{\frac{1}{2}} \left(49x^2\right)^{\frac{1}{2}}$
$\left(6x^{\frac{5}{2}}\right)\left(7x^{\frac{2}{2}}\right) = 42x^{\frac{7}{2}}$

20. $\sqrt[3]{8x^3y^6} \cdot \sqrt{49x^2y^{10}} =$ $\left(2xy^2\right)\left(7xy^5\right)$
$= 14x^2y^7$

Example 2: Solve for x.

a) $4^x = 64$

b) $4^{2x+1} = 16^{-3}$

c) $9^{x+1} = 27$