Unit 1: Polynomial Expressions and Functions

1.1 Dividing a Polynomial by a Binomial

Recall: Long Division

1. $2748 \div 13$

$$\begin{array}{r} 211 \\ 13\overline{)2748} \\ -26 \\ \hline 14 \\ -13 \\ \hline 18 \\ -13 \\ \hline 5 \end{array}$$

2. $12345 \div 23$

$$\begin{array}{r} 536 \\ 23\overline{)12345} \\ -115 \\ \hline 84 \\ -69 \\ \hline 155 \\ -138 \\ \hline 17 \end{array}$$

Dividing a polynomial.

1. Divide

$$\begin{array}{r} 2x^2+5x+3 \\ x-1\overline{)2x^3+3x^2-2x+5} \\ -(2x^3-2x^2) \\ \hline 5x^2-2x \\ -(5x^2-5x) \\ \hline 3x+5 \\ -(3x-3) \\ \hline 8 \end{array}$$

$\therefore 2x^3+3x^2-2x+5$
$= (x-1)(2x^2+5x+3)+8$

Division Statement: For $P(x) = (x - a)\,Q(x) + R$, where $P(x)$ is the original polynomial, $(x - a)$ is the binomial divisor, $Q(x)$ is the quotient polynomial (which has a degree 1 less than $P(x)$, and R is the remainder (which is a constant).

Example 2. Divide $(-4x^4 + 2x^2 - x - 3) \div (x - 3)$

* add a zero placeholder for x^3 term

$$
\begin{array}{r}
-4x^3 - 12x^2 - 34x - 103 \\
x-3 \overline{\big)\, -4x^4 + 0x^3 + 2x^2 - x - 3} \\
-(-4x^4 + 12x^3) \\
\hline
-12x^3 + 2x^2 \\
-(-12x^3 + 36x^2) \\
\hline
-34x^2 - x \\
-(-34x^2 + 102x) \\
\hline
-103x - 3 \\
-(-103x + 309) \\
\hline
-312
\end{array}
$$

$-4x^4 + 2x^2 - x - 3$
$= (-4x^3 - 12x^2 - 34x - 103)(x - 3)$
-312

Example 3: Divide.

$$x + 2\overline{\big)\,3x^4 - x^3 + 3x - 20}$$

$$
\begin{array}{r}
3x^3 - 7x^2 + 14x - 25 \\
x+2 \overline{\big)\, 3x^4 - x^3 + 0x^2 + 3x - 20} \\
-(3x^4 + 6x^3) \\
\hline
-7x^3 + 0x^2 \\
-(-7x^3 - 14x^2) \\
\hline
14x^2 + 3x \\
-(14x^2 + 28x) \\
\hline
-25x - 20 \\
-(-25x - 50) \\
\hline
30
\end{array}
$$

$3x^4 - x^3 + 3x - 20$
$= (x + 2)(3x^3 - 7x^2 + 14x - 25)$
$+ 30$

Synthetic Division

- Synthetic Division can be used to divide a polynomial by a binomial of the form $\boxed{(x-a),}$ a ϵ Integers.
- the variables are removed and only the coefficients are recorded.

Example 1: $(5x^2+7x-4) \div (x-2)$

Compare $\underline{x-a}$ and $\underline{x-2}$. Therefore, $a=2$.

$$\begin{array}{r|rrr} 2 & 5 & 7 & -4 \\ & \downarrow & 10 & 34 \\ \hline & 5 & 17 & \boxed{30} \end{array}$$

$5x+17$ R. 30

$5x^2+7x-4=(x-2)(5x+17)+30$

Write the value of a on the left. Write the coefficients of the polynomial on the right. Bring down the first coefficient, 5.

Multiply 5 by the value of a, 2. Record the product, 10, beneath the second coefficient, 7, then add.

Multiply the sum, 17, by the value of a, 2. Record the product, 34, beneath the third coefficient, -4, then add.

The numbers below the line are the coefficients of the quotient polynomial and the remainder. Since the dividend is a polynomial of degree 2, the quotient is a polynomial of degree 1.

Example 2: $(3x^4-x^3+3x-20) \div (x+2)$

Sub in $0x^2$

$x+2$

$a=-2$

$$\begin{array}{r|rrrrr} -2 & 3 & -1 & 0 & 3 & -20 \\ & \downarrow & -6 & 14 & -28 & 50 \\ \hline & 3 & -7 & 14 & -25 & \boxed{30} \end{array}$$

$3x^3-7x^2+14x-25$ R. 30

$3x^4-x^3+3x-20=(x+2)(3x^3-7x^2+14x-25)+30$

Example 3: $(-3x^4 + 2x^3 + 3x^2 - 4x + 5) \div (x + 2)$

$$\begin{array}{r|rrrrr} -2 & -3 & 2 & 3 & -4 & 5 \\ & \downarrow & 6 & -16 & 26 & -44 \\ \hline & -3 & 8 & -13 & 22 & \boxed{-39} \end{array}$$

$-3x^3 + 8x^2 - 13x + 22$ R. -39

*Use long division

Example 4: $(6x^3 + 4x^2 + 8) \div \underline{(2x + 4)}$

$$\begin{array}{r|l} & 3x^2 - 4x + 8 \\ \hline 2x+4 & 6x^3 + 4x^2 + 0x + 8 \\ & -(6x^3 + 12x^2) \\ \hline & \quad -8x^2 + 0x \\ & -(-8x^2 - 16x) \\ \hline & \qquad 16x + 8 \\ & -(16x + 32) \\ \hline & \qquad -24 \end{array}$$

$6x^3 + 4x^2 + 8 = (3x^2 - 4x + 8)(2x + 4) - 24$

Factoring Polynomials

Recall: Trinomial Factoring

1. $6x^2 + 7x - 3$

-18

$6x^2 + 9x - 2x - 3$

$3x(2x+3) - 1(2x+3)$

$(3x-1)(2x+3)$

2. $2x^3 - 3x^2 - 14x$

$x(2x^2 - 3x - 14)$

$x(2x-7)(x+2)$

3. $49x^2 - 36y^4$

$(7x + 6y^2)(7x - 6y^2)$

Recall: $(5x^3 - 2x^2 + 8x - 1) \div (x - 2)$

$$\begin{array}{r|rrrr} 2 & 5 & -2 & 8 & -1 \\ & \downarrow & 10 & 16 & 48 \\ \hline & 5 & 8 & 24 & \boxed{47} \end{array}$$

$5x^2 + 8x + 24 \quad R.47$

Note the divisor is $x - 2$.

Let $P(x) = 5x^3 - 2x^2 + 8x - 1$

Evaluate the polynomial for $x = 2$.

$$P(2) = 5(2^3) - 2(2^2) + 8(2) - 1$$
$$= 40 - 8 + 16 - 1$$
$$= 47$$

The value of the polynomial when $x = 2$ is equal to the remainder when the polynomial is divided by $x - 2$. This is called **Remainder Theorem**.

Remainder Theorem: When a polynomial, $P(x)$, is divided by $x - a$, a ϵ Integers, the remainder is $P(a)$.

Example 1: Determine the remainder when $2x^4 - 5x^3 - 5x^2 + 5x + 3$ is divided by each binomial:

a) $x - 3$

$x = 3$

$f(3) = 2(3)^4 - 5(3)^3 - 5(3)^2 + 5(3) + 3$

$= 162 - 135 - 45 + 15 + 3$

$= 0$ ∴ You can divide the polynomial evenly by x-3

b) $x + 2$

$x = -2$

$f(-2) = 2(-2)^4 - 5(-2)^3 - 5(-2)^2 + 5(-2) + 3$

$= 32 + 40 - 20 - 10 + 3$

$= 45$

Factor Theorem: For a ϵ Integers, $x - a$ is a factor of the polynomial $P(x)$ if $P(a) = 0$.

Example 2: Which binomials are factors of $x^3 - 6x^2 + 5x + 12$?

a) $x + 1$

$x = -1$

$(-1)^3 - 6(-1)^2 + 5(-1) + 12$

$-1 - 6 - 5 + 12 = 0$

∴ x+1 is a factor

b) $x - 3$

$x = 3$

$(3)^3 - 6(3)^2 + 5(3) + 12$

$= 27 - 54 + 15 + 12$

$= 0$

c) $x + 4$

$x = -4$

$(-4)^3 - 6(-4)^2 + 5(-4) + 12$

$= -64 - 96 - 20 + 12$

$= -168$

d) $x - 4$

$x = 4$

$(4)^3 - 6(4)^2 + 5(4) + 12$

$= 64 - 96 + 20 + 12$

$= 0$

Factor Property: If $x - a$, $a \in$ Integers, is a factor of a polynomial, then a is a factor of the constant term in the polynomial.

$\pm 1, \pm 2, \pm 3, \pm 6$

$P(x)$

Example 3: Factor Fully: $2x^3 - 9x^2 + 7x \boxed{+ 6}$

Try $x+1$
$x = -1$
$2(-1)^3 - 9(-1)^2 + 7(-1) + 6$
$-2 - 9 - 7 + 6 \neq 0$

Try $x-3$.
$x = 3$
$2(3)^3 - 9(3)^2 + 7(3) + 6$
$= 54 - 81 + 21 + 6$
$= 0$ ✓

$x-3$ is a factor.

$$\begin{array}{r|rrrr} 3 & 2 & -9 & 7 & 6 \\ & \downarrow & 6 & -9 & -6 \\ \hline & 2 & -3 & -2 & \boxed{0} \end{array}$$

$(x-3)(2x^2-3x-2) = P(x)$
$(x-3)(2x+1)(x-2) = P(x)$

$P(x)$ $\pm1, \pm2$

Example 4: Factor Fully: $3x^3 - 4x^2 - 5x + 2$

Try $x+1$.

$$\begin{array}{r|rrrr} -1 & 3 & -4 & -5 & 2 \\ & \downarrow & -3 & 7 & -2 \\ \hline & 3 & -7 & 2 & \boxed{0} \end{array}$$

$P(x) = (x+1)(3x^2 - 7x + 2)$

$P(x) = (x+1)(3x-1)(x-2)$

Hwk pg 124 #6a, b, d
pg 133 #2, 4a, b, 5

1.3 Graphing Polynomial Functions

Recall: Graphs of quadratic functions.

For the quadratic function below, graph it, then identify the coordinates of the vertex, the domain, the range, the direction of opening and the intercepts.

1. $y = -x^2 - 4x + 5$ Leading Coeffecient $= -1$, opens down

Vertex

$x = \frac{-b}{2a} = \frac{-(-4)}{2(-1)}$

$= \frac{4}{-2} = -2$

$y = -(-2)^2 - 4(-2) + 5$

$= -4 + 8 + 5$

$= 9$

$(-2, 9)$

y-int

$y = 5$

x-int

$0 = -x^2 - 4x + 5$

$0 = x^2 + 4x - 5$

$0 = (x+5)(x-1)$

$x + 5 = 0$ $x - 1 = 0$

$x = -5$ $x = 1$

Domain

$(-\infty, \infty)$

$\{x \mid x \in \mathbb{R}\}$

Range

$(-\infty, 9]$

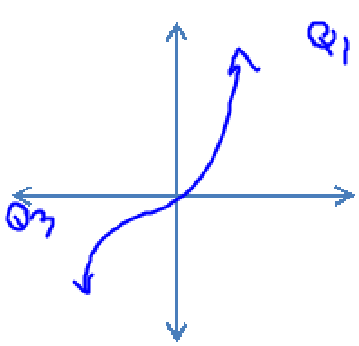
$\{y \mid y \leq 9\}$

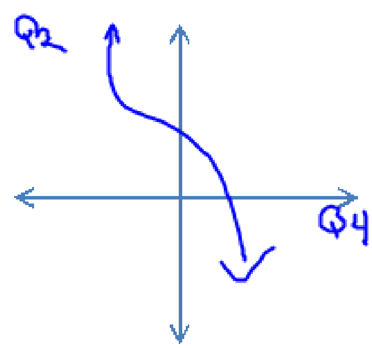
Graphing Polynomial Functions

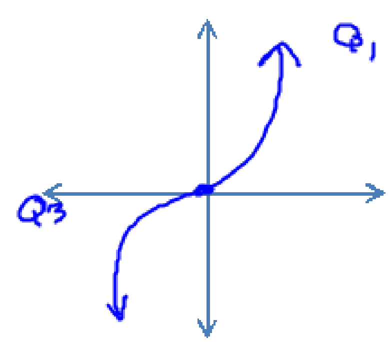
Recall: The **degree** of a polynomial is the largest exponent value.

Investigate the polynomial graphs of degree 3. These are cubic graphs. Sketch each graph below.

a. $y = 2x^3 + 4x$ b. $y = -4x^3 + 2x^2 + 3$ c. $y = x^3$





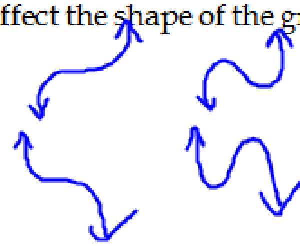


Graph	Number of x-intercepts	Number of hills	Number of valleys	y-intercept
$y = 2x^3 + 4x$	1	0	0	0
$y = -4x^3 + 2x^2 + 3$	1	1	1	3
$y = x^3$	1	0	0	0

- How does the sign of the cubic term affect the shape of the graph?

  Positive → Q3 to Q1

  Negative → Q2 to Q4



- What do you notice about the y-intercept for each equation?

  The y-intercept is the constant (the # that doesn't change)

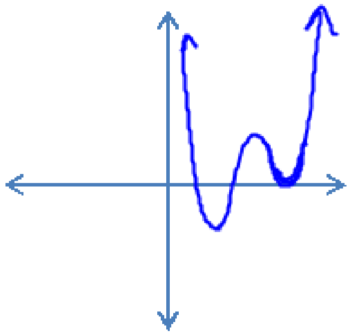
- What generalizations can you make about cubic functions?

  At most 1 hill + 1 valley

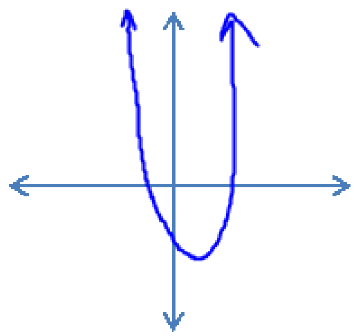
B. Degree 4, or Quartic graphs.

Sketch each graph.

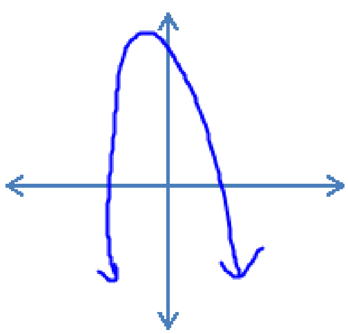
a. $y = x^4 - 11x^3 + 42x^2 - 64x + 32$



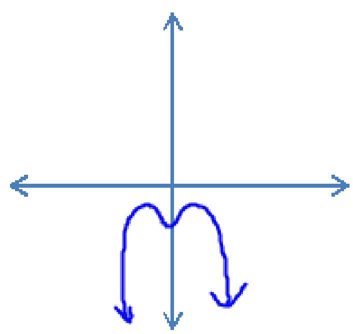
b. $2x^4 - 6x - 4$



c. $y = -x^4 + 3x^3 - 4x^2 + 7$

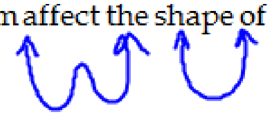


d. $-2x^4 + 3x^2 - 5$



- How does the sign of the x^4 term affect the shape of the graph?

positive: $Q_2 \Rightarrow Q_1$



negative: $Q_3 \Rightarrow Q_4$

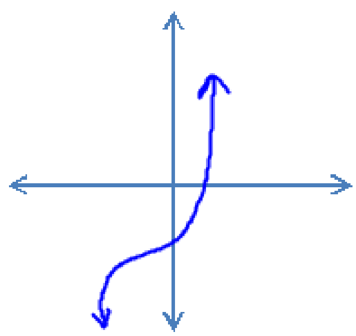
- How does the value of the constant term affect the graph of the function?

Constant = y-intercept

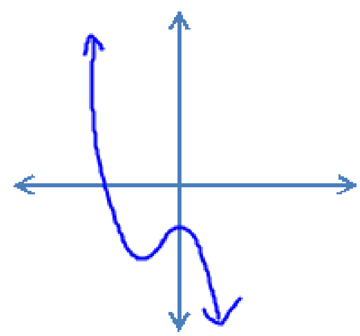
C. Degree 5, or Quintic graphs.

Sketch each graph.

a. $y = 3x^5 - 6$

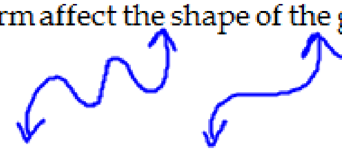


b. $y = -3x^5 - 5x^2 + 3x - 4$



- How does the sign of the x^5 term affect the shape of the graph?

Positive: $Q_3 \Rightarrow Q_1$



Negative: $Q_2 \Rightarrow Q_4$

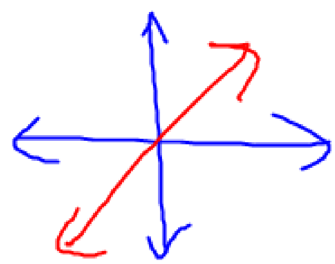
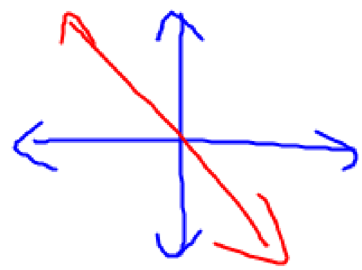
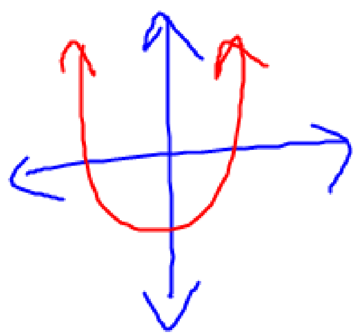
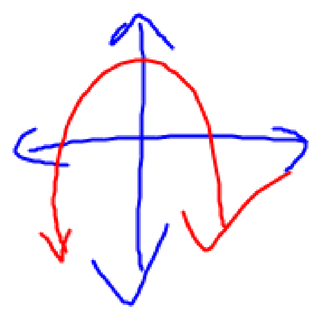
- How does the value of the constant term affect the graph?

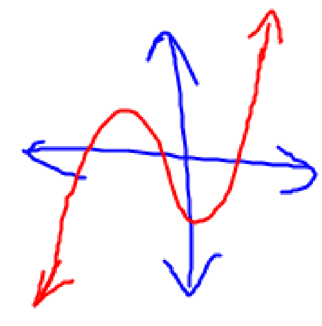
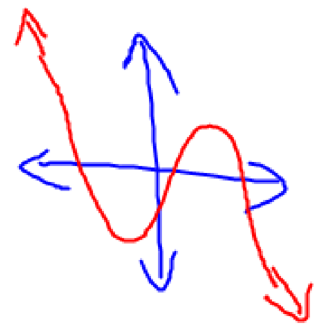
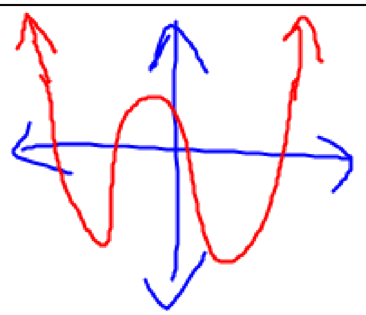
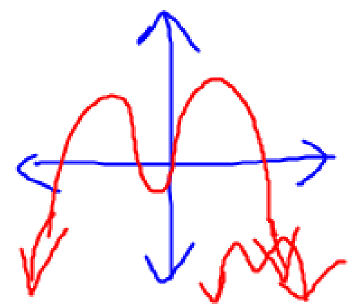
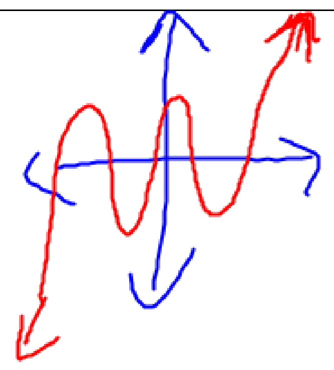
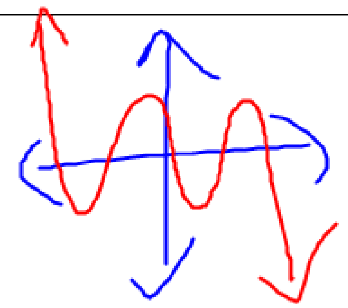
Polynomial Functions: A polynomial function of degree n can be written in standard form as

$f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x^1 + a_0$ where n is a whole number and a_n, a_{n-1}, a_{n-2} … are all real numbers.

The coefficient of the highest power of x is the leading coefficient.

A polynomial function can be described by its degree and whether the degree is even or odd.

Degree	Type of polynomial function	Positive leading coefficient	Negative leading coefficient
1	Linear Odd-degree y=ax+b		
2	Quadratic Even-degree		

3	Cubic Odd- degree		
4	Quartic Even-degree		
5	Quintic Odd-degree		

To graph a polynomial function:

A. Consider the ***end behaviour*** of the graph.

- **Odd-degree polynomial functions**:

Leading Coefficient	End behaviour of the graph
Positive	As $x \to \infty$, the graph rises to the right. As $x \to -\infty$, the graph falls to the left.
Negative	As $x \to \infty$, the graph falls to the right. As $x \to -\infty$, the graph rises to the left.

- **Even-degree polynomial functions**

Leading Coefficient	End behaviour of the graph
Positive	As x→∞, the graph rises to the right. As x→−∞, the graph rises to the left.
Negative	As x→∞, the graph falls to the right. As x→−∞, the graph falls to the left.

B. Find the ***x-intercepts*** (or zeros) of the graph by factoring.

- Since the y-value of a polynomial function can only change from negative to positive or positive to negative at a zero, use a sign diagram to determine the sign of the function over each interval.

Example: Here is a factored polynomial. Complete a sign diagram to help with the graph.

Label all x- and y-intercepts.

$f(x) = (x-1)(x-2)(x+3)(x+1)$

Quartic

because your leading coefficient will be positive.

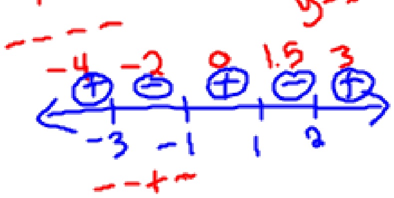
x ints

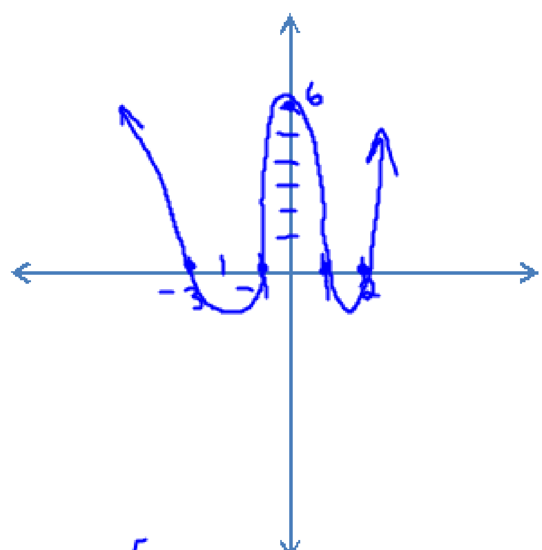
$\{1, 2, -3, -1\}$

y int

$y = (-1)(-2)(3)(1)$

$y = 6$





C. Find the y-intercept and graph it.

D. The point where the graph changes from increasing to decreasing is called a **local maximum point.** The y-value of this point is greater than those of neighboring points.

The point where the graph changes from decreasing to increasing is called a **local minimum point.** The y-value of this point is less than those of neighboring points.

E. A polynomial of degree n can have at most n x-intercepts, and at most $(n-1)$ local maximum or minimum points.

F. The graph of a polynomial function is smooth and continuous, which means it has no sharp corners.

Graph the following polynomial functions:

Example 1. $h(x) = x^3 + x^2 - 6x$

y int

0

x int → factor

$x(x^2+x-6) = 0$

$x(x+3)(x-2) = 0$

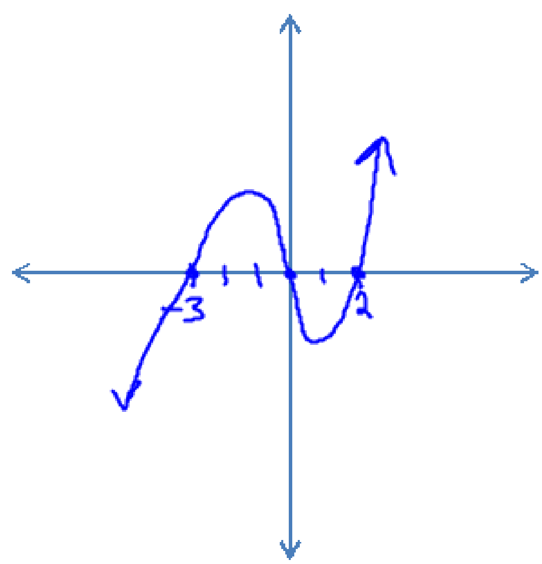
$\therefore$ x ints $= \{2, -3, 0\}$

$-\ -\ -\quad +\ +\ -$

$-4 \quad -1 \quad 1 \quad 3$

$-3 \quad 0 \quad 2$

$-\ +\ -$



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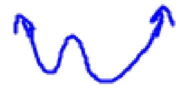
Example 2: $f(x) = 2x^4 - x^3 - 14x^2 + 19x - 6$ $\pm 1, \pm 2, \pm 3, \pm 6$

y int $= -6$

Quartic + positive



Factor

$(x-1)$

1	2	−1	−14	19	−6
	↓	2	1	−13	6
	2	1	−13	6	0

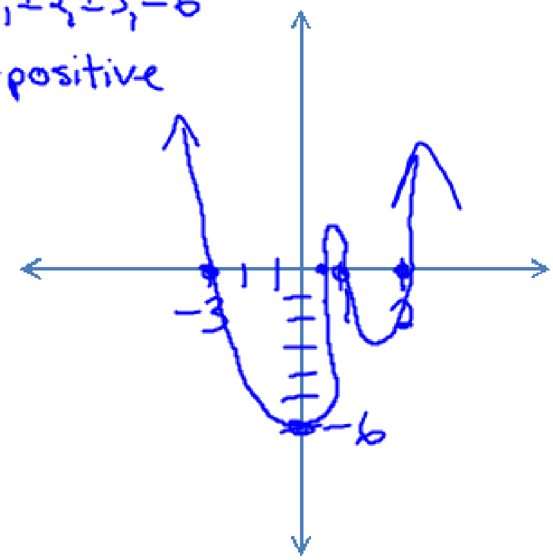
$(x-1)(2x^3 + x^2 - 13x + 6)$

2	2	1	−13	6
	↓	4	10	−6
	2	5	−3	0

$(x-1)(x-2)(2x^2 + 5x - 3)$

$(x-1)(x-2)(2x-1)(x+3)$

$x = 1, 2, \frac{1}{2}, -3$



Multiplicity of a Zero.

The multiplicity of a zero refers to a repeated root. That is, the same root may occur more than once. For example:

$x^2 - 2x + 1 = 0$ can be written as $(x-1)^2 = 0$. Therefore the root of $\boxed{x = 1}$ has a multiplicity of 2. So the related function has a zero of multiplicity $\boxed{2.}$

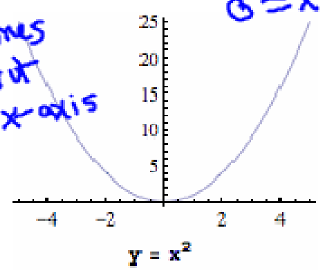
$x^3 - 3x^2 + 3x - 1 = 0$ can be written as $(x-1)^3 = 0$. Therefore the root of $x = 1$ has a multiplicity of 3. So the related function has a zero of multiplicity $\boxed{3.}$

*Note the behaviour of the graph at the zero in each case.

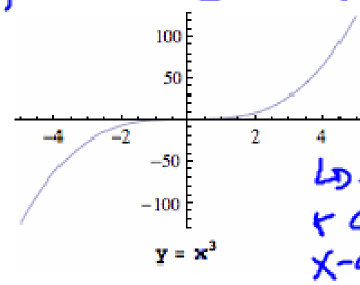
Even multiplicity

2 or 4 times
↳ flatten out
& touch x-axis

$y = x(x)$
$0 = x(x)$



$y = x(x)(x)$
$0 = x(x)(x)$



Odd
3, 5 times
↳ flatten out
& crosses
x-axis

Both graphs have an x-intercept of 0. The graph of $f(x) = x^2$ touches the x-axis but does not cross the axis at this point. However, the graph of $f(x) = x^3$ does cross at this point.

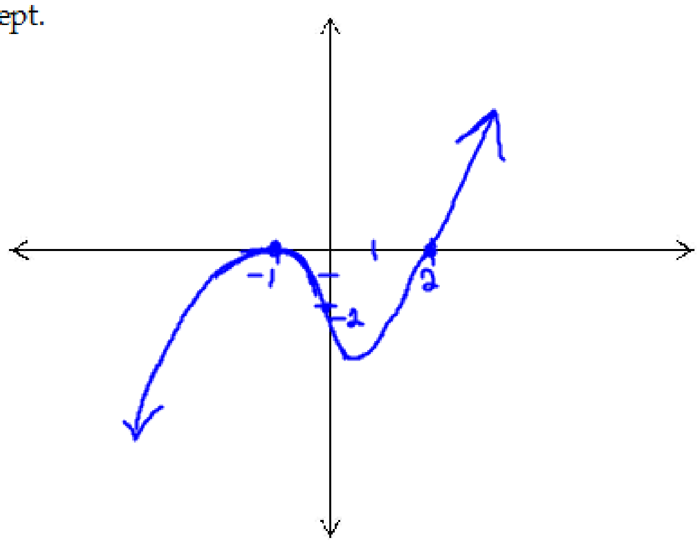
The difference in behaviour is related to the multiplicity of the zero. In general, if the multiplicity is even, the graph touches the x-axis at the related x-intercept, but does not cross. If the multiplicity is odd, then the graph crosses the x-axis at the related x-intercept but levels out at the x-intercept.

Example 1: $f(x) = (x+1)^4(x-2)$

x ints
−1 (multiplicity of 4)
2

y int
$y = (1)^4(-2)$
$y = -2$

+ −   + +
−2   0   3
−1   2
⊖   ⊖   ⊕



Example 2: $g(x) = -(x+1)^3(x-3)$

x ints

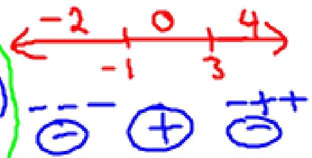
-1 (multiplicity of 3)

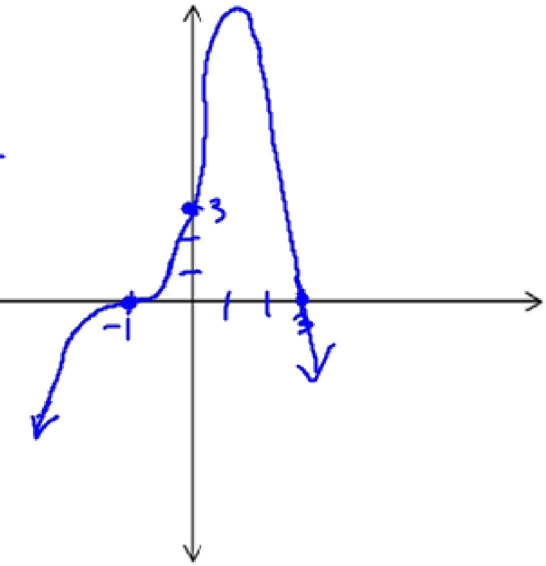
3

y int

$y = -(1)^3(-3)$

$y = 3$





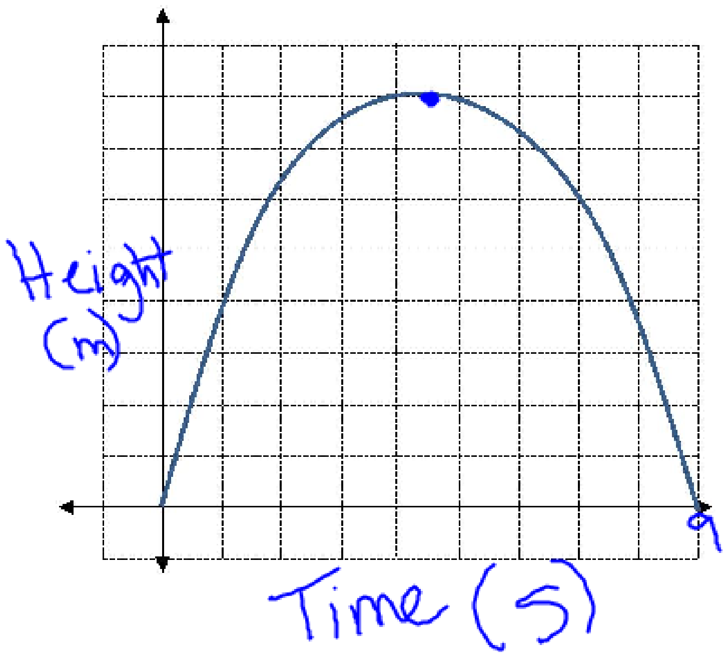
Modelling and Solving Problems with Polynomial Functions

Watch: Pumpkin Chuckin' from Discovery Learning.

This graph represents the path of one of the pumpkins as it is launched into the air.

What is the maximum height?

8 m



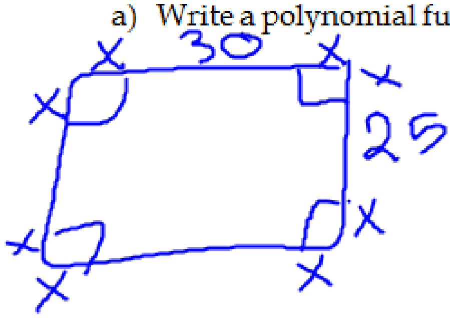
How ~~far~~ did the pumpkin travel?

long

9s

Example 1: A piece of cardboard 30 cm long and 25 cm wide is used to make a box with no lid. Equal squares of side length x centimetres are cut from the corners and the sides are folded up.

a) Write a polynomial function to represent the volume, V, of the box in terms of x.



$V = lwh$

$V = (30-2x)(25-2x)x$

$l = 30-2x$

$w = 25-2x$

$h = x$

b) Graph the function by hand. Examine the graph.

$V=(30-2x)(25-2x)(x)$

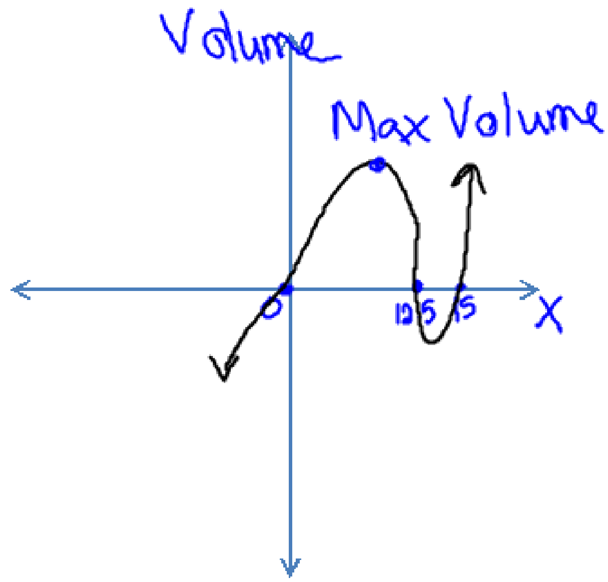
x ints
0, 12.5, 15

y int
0

$30-2x=0$
$\frac{30}{2}=\frac{2x}{2}$
$x=15$

$25-2x=0$
$25=2x$
$x=12.5$

Cubic
$(-2x)(-2x)(x)$
$4x^3$



c) Graph the function using a calculator. What is the domain?

(sketch the window from your calculator below. Include the values from your window)

Window

Xmin
Xmax
Xscl
Ymin
Ymax
Yscl
Xres

d) To the nearest cubic cm, what is the maximum volume of the box? What size should be cut out to create a box with this volume? To the nearest tenth of a cm, what are the dimensions of the box?

Example 2:

Bill is preparing to make an ice sculpture. He has a block of ice that is 3 ft. wide, 4 ft. high, and 5 ft. long. Bill wants to reduce the size of the block of ice by removing the same amount from each of the three dimensions. He wants to reduce the volume of the ice block to 24 ft³.

a. Write a polynomial function to model this situation.

Let x = amount removed

$l = 5 - x \quad w = 3 - x \quad h = 4 - x$

$V = (5-x)(3-x)(4-x)$

b. How much should he remove from each ~~dimension~~?

$24 = (5-x)(3-x)(4-x)$ Find x.

$0 = (15 - 5x - 3x + x^2)(4-x) - 24$

$0 = (15 - 8x + x^2)(4-x) - 24$

$0 = (60 - 32x + 4x^2 - 15x + 8x^2 - x^3) - 24$

$0 = -x^3 + 12x^2 - 47x + 36$

$0 = x^3 - 12x^2 + 47x - 36$ Factor

$x = 1$

$$\begin{array}{r|rrrr} 1 & 1 & -12 & 47 & -36 \\ & \downarrow & 1 & -11 & 36 \\ \hline & 1 & -11 & 36 & \boxed{0} \end{array}$$

$0 = (x-1)(x^2 - 11x + 36) \rightarrow$ Check discriminant → We can't factor.

$\sqrt{b^2 - 4ac}$

$\sqrt{(-11)^2 - 4(1)(36)}$

$\sqrt{-23}$

∴ Our only solution is $x = 1$.

He should remove 1 ft. from each side.