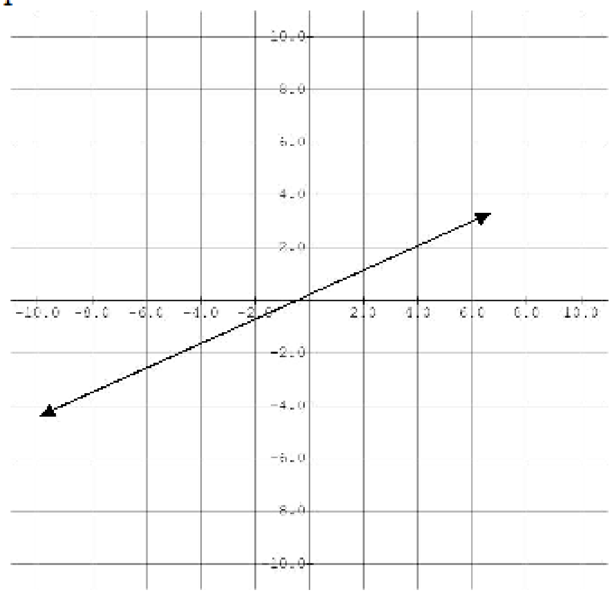
Unit 2: Radical and Rational Functions

2.1 Properties of Radical Functions

Recall: Domain and Range.

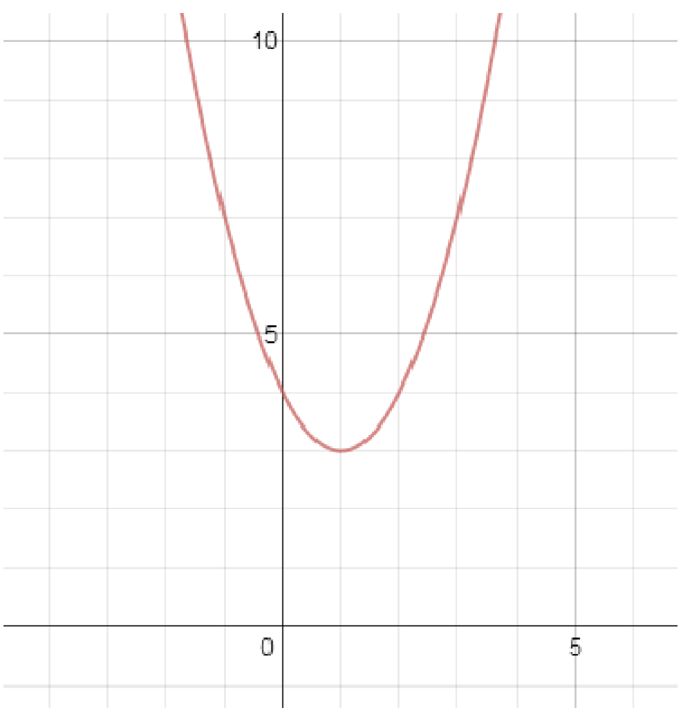
Example 1:



Domain:
$(-\infty, \infty)$
$x \in \mathbb{R}$

Range:
$(-\infty, \infty)$
$x \in \mathbb{R}$

Example 2:



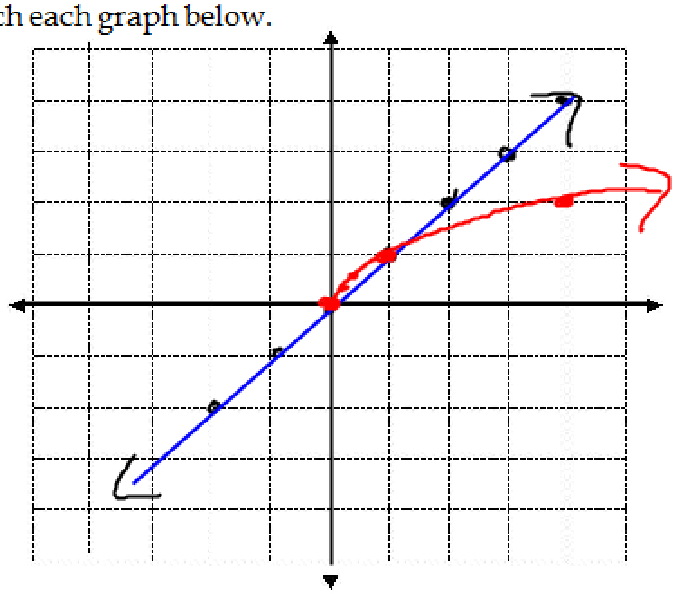
Domain:
$(-\infty, \infty)$
$x \in \mathbb{R}$

Range:
$[3, \infty)$
$\{y \mid y \geq 3\}$

Complete the table of values for $y = x$ and $y = \sqrt{x}$.

x	-4	-2	-1	0	$\frac{1}{9}$	$\frac{1}{4}$	1	4	9
$y = x$	-4	-2	-1	0	$\frac{1}{9}$	$\frac{1}{4}$	1	4	9
$y = \sqrt{x}$				0	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3

Sketch each graph below.



$y = \sqrt{x}$

Domain: $[0, \infty)$

$\{x \mid x \geq 0\}$

Range: $[0, \infty)$

$\{y \mid y \geq 0\}$

A **Radical Function** has the form $y = \sqrt{f(x)}$, where $f(x)$ is the function. The square root of a number is only defined for non-negative numbers, so the domain of $y = \sqrt{f(x)}$ is the set of values of x for which $f(x) \geq 0$.

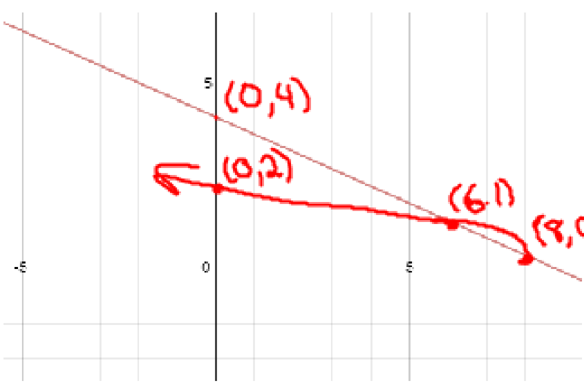
When graphing, use these facts:

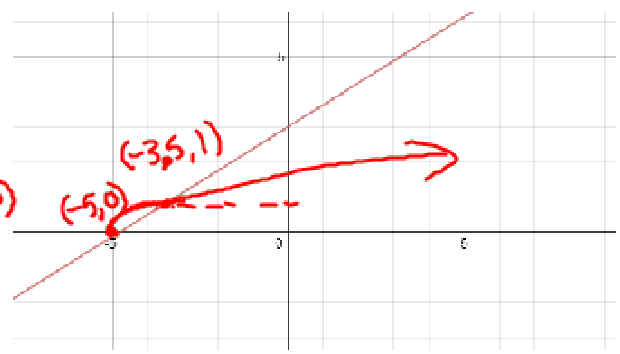
- If 0 and 1 are in the range of $f(x)$, then the y-coordinates corresponding to these x values lie on both graphs; these are ***invariant points***.
- Where the graph of $y = f(x)$ lies between $y = 1$ and the x-axis, the graph of $y = \sqrt{f(x)}$ lies above the graph of $f(x)$
- Where the graph of $y = f(x)$ lies above $y = 1$, the graph of $y = \sqrt{f(x)}$ lies below the graph of $f(x)$
- Where the graph of $y = f(x)$ lies below the x-axis, the graph of $y = \sqrt{f(x)}$ does not exist.

Example 1: For each graph of $y = f(x)$,

a. Sketch the graph of $y = \sqrt{f(x)}$

b. State the domain and range of $y = \sqrt{f(x)}$.





D: $(-\infty, 8]$

R: $[0, \infty)$

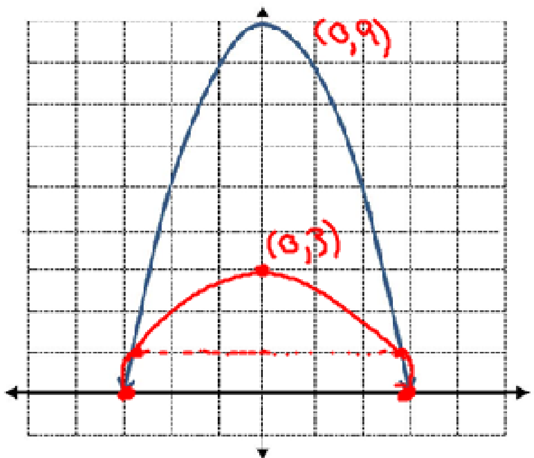
D: $[-5, \infty)$

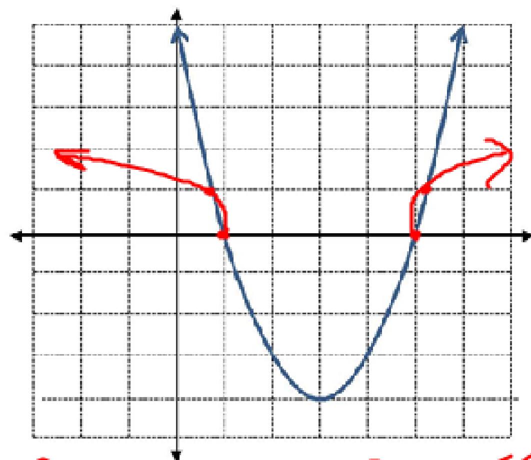
R: $[0, \infty)$

Example 2: For the graph of each quadratic function $y = f(x)$ below,

a. Sketch the graph of $y = \sqrt{f(x)}$.

b. State the domain and range of $y = \sqrt{f(x)}$.





D: $[-3, 3]$ $\{x \mid -3 \leq x \leq 3\}$

R: $[0, 3]$

$\{y \mid 0 \leq y \leq 3\}$

D: $(-\infty, 1] \cup [5, \infty)$

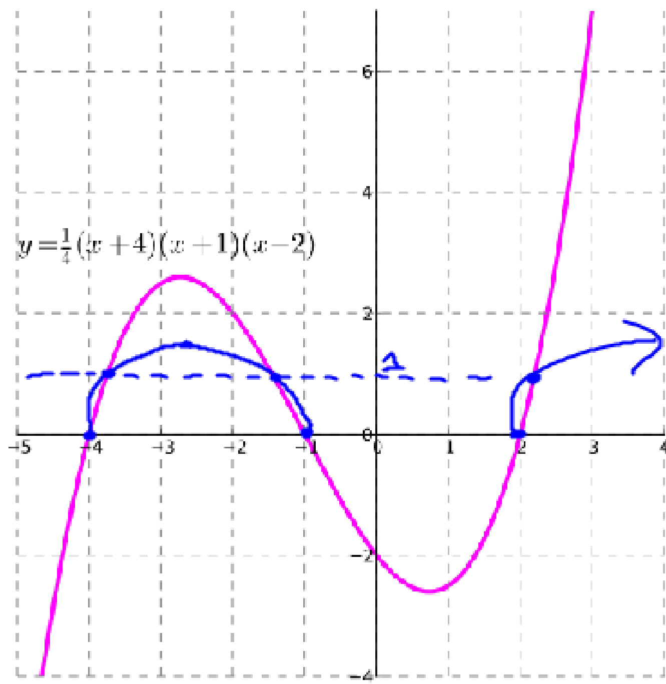
$\{x \mid x \leq 1, x \geq 5\}$

R: $[0, \infty)$

Example 3: For the graph of the cubic function $y = f(x)$ below,

a. Sketch the graph of $y = \sqrt{f(x)}$.

b. State the domain and range of $y = \sqrt{f(x)}$.



D: $[-4,-1] \cup [2,\infty)$

R: $[0,\infty)$

pg 86 #2, 3, 4

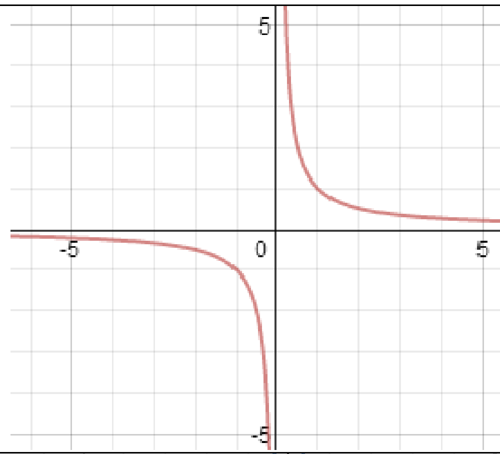
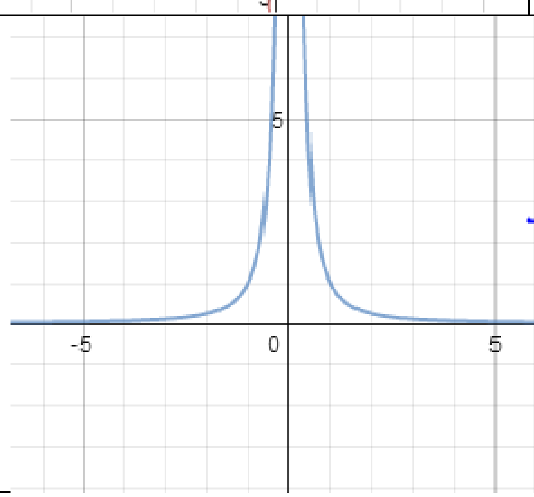
2.2 Graphing Rational Functions

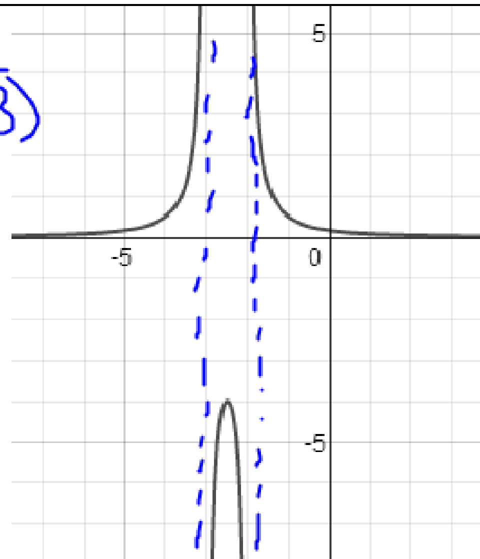
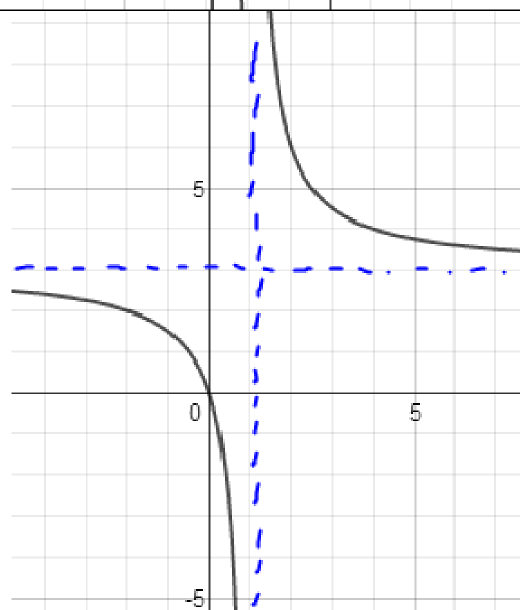
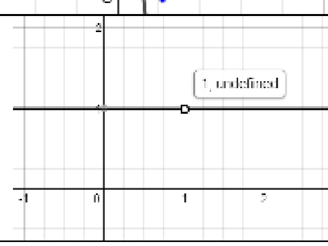
A ***rational function*** is a function of the form $f(x) = \frac{p(x)}{q(x)}$ where $q(x) \neq 0$ and where p(x) and q(x) are polynomials in x.

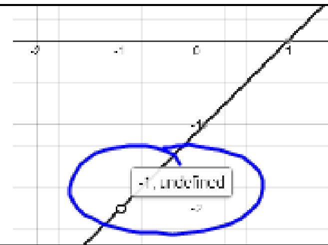
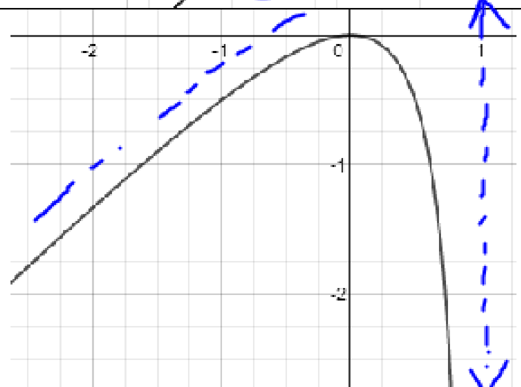
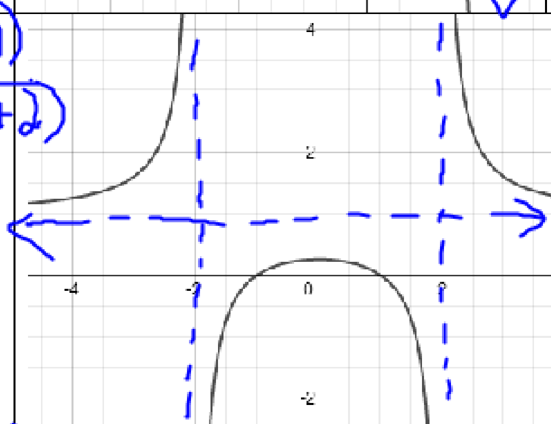
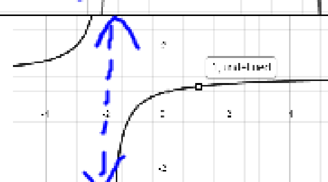
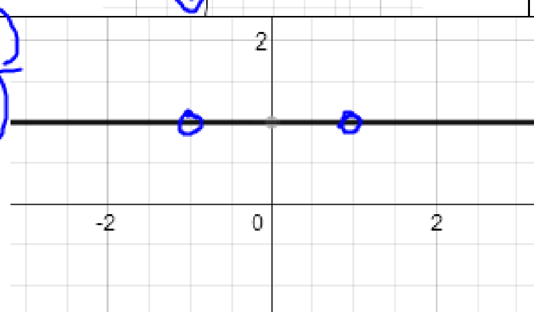
Some examples of rational functions are:
$f(x) = \frac{1}{2x+1}$ and $g(x) = \frac{x^2+1}{(x-1)(x+5)}$

When you have a rational function, you need to be careful not to divide by zero! Complete the following table to see what can happen when you divide by zero.

Denominator $= 0$ → Vertical Asymptote or hole

Function	Function in Factored Form	Graph	Non-permissible Values of x
$y = \frac{1}{x}$	$y = \frac{1}{x}$		$x=0$ V.A @ $x=0$ H.A. @ $y=0$
$y = \frac{1}{x^2}$	$y = \frac{1}{(x)(x)}$		$x=0$ V.A. @ $x=0$ H.A. @ $y=0$

$y=\frac{1}{x^2+5x+6}$	$y=\frac{1}{(x+2)(x+3)}$		V.A. @ $x=-2$ and $x=-3$ H.A @ $y=0$
$y=\frac{3x}{x-1}$			VA when $x=1$ H.A. when $y=3$
$y=\frac{x-1}{x-1}$	$x=1$ $\frac{1-1}{1-1}=\frac{0}{0}$		Hole $x=1$

$y = \frac{x^2-1}{x+1}$	$y=\frac{(x+1)(x-1)}{x+1}$		$x = -1$ ↳ Hole
$y = \frac{x^2}{x-1}$			V.A. @ $x = 1$
$y = \frac{x^2-1}{x^2-4}$	$y=\frac{(x-1)(x+1)}{(x-2)(x+2)}$		V.A. @ $x = 2, -2$ H.A. @ $y = 1$
$y = \frac{x^2-1}{x^2+x-2}$	$y=\frac{(x+1)(x-1)}{(x+2)(x-1)}$		$x=-2$ → V.A. $x = 1$ → Hole
$y = \frac{x^2-1}{x^2-1}$ **The asymptotes/holes are missing from the graph – can you put them in where they should be?**	$y=\frac{(x-1)(x+1)}{(x-1)(x+1)}$		$x = 1$ & -1 ↳ Holes

Note:

- For rational functions, how can you determine the non-permissible values of x?

The non-permissible values occur when the denominator equals zero.
→ Hole
→ Asymptote

- How can you tell whether the graph of a rational function has an asymptote or a hole at a non-permissible value of x?

Asymptote occurs when your factor is <u>only</u> in the denominater.
A hole occurs when you have the same factor in the numerator & denominater.

- How can you tell whether a graph of a rational function has a horizontal asymptote?

① When the degree of the num. and denom. are the same, there is a H.A. at $y=\frac{a}{b}$ (coefficients of terms w/ highest exponents).

② When the degree of the denom. is larger than the num., there is a H.A. at $y=0$.

③ When the degree of the num. is larger, there is an oblique asymptote.

2.3 Analyzing Rational Functions

A *Rational Function* has the form $y = \frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials and $g(x) \neq 0$.

Note: The characteristics of the graph of a rational function can be determined from an equation of the function.

Characteristic #1: Non-permissible values of x.

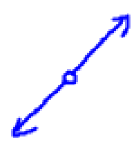
For each rational function,

- Identify the non-permissible values of x.
- Describe the behaviour of the graph at these values…is there an asymptote or a hole?

Example 1: $y = \frac{x+3}{(x-3)(x+1)}$

$x \neq 3$ and $x \neq -1$

$\therefore$ We have 2 V.A. @ $x=3$ and $x=-1$.



Example 2: $y = \frac{\cancel{(x-3)}(x+1)}{\cancel{x-3}}$ $y = x+1$

$x \neq 3$ $\therefore$ There is a hole at $x=3$.

Example 3: $y = \frac{x^2}{4-x^2} = \frac{x^2}{-(x^2-4)} = \frac{x^2}{-(x-2)(x+2)}$

$x \neq 2$ and $x \neq -2$

V.A @ $x=2$ and $x=-2$

Example 4: $y = \frac{x^2-25}{x+5} = \frac{(x-5)(x+5)}{(x+5)}$

$x \neq -5$

$\therefore$ Hole at $x=-5$

Characteristic #2: Horizontal and Oblique Asymptotes

Example 1: $y = -\frac{4}{x+4}$

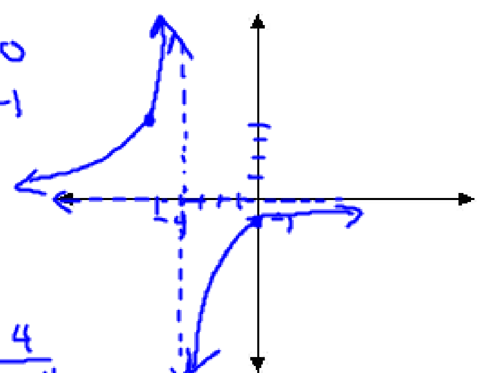
$x \neq -4$

V.A. @ $x=-4$

Find y-int. $x=0$

$y = \frac{-4}{0+4} = \frac{-4}{4} = -1$

$\therefore (0,-1)$



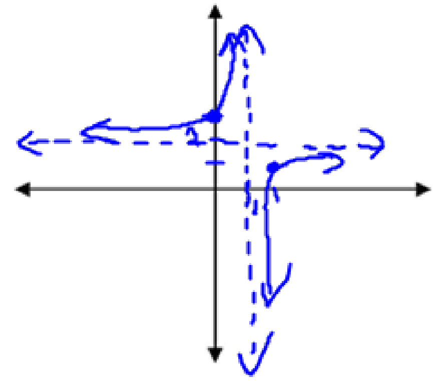
The degree of the expression in the numerator is less than the degree of the expression in the denominator.

So, as $x \to \infty$, $y \to 0$, and the graph has a horizontal asymptote at $\boxed{y = 0}$.

$x=-5$ $\quad y = \frac{-4}{-5+4} = \frac{-4}{-1} = 4$

$\therefore (-5,4)$

Example 2: $y = \frac{2x-3}{x-1}$ $\frac{2}{1} = 2$ $x=1 \to$ V.A.



The expression in the numerator and the denominator have the same degree. Using synthetic division to divide:

y int $\to$ $x=0$

$y = \frac{2(0)-3}{0-1} = \frac{-3}{-1} = 3$

$\therefore (0,3)$

$$\begin{array}{r|rr} 1 & 2 & -3 \\ & & 2 \\ \hline & 2 & -1 \end{array}$$

The quotient is 2 and the remainder is -1. The remainder can be written as a fraction of the divisor, so $y = \frac{2x-3}{x-1}$ can be written as $\boxed{y = 2 - \frac{1}{x-1}}$.

So, as $|x| \to \infty, \frac{1}{x-1} \to 0$, so $y \to 2 - 0$, or 2, and the graph has a horizontal asymptote at $\boxed{y = 2.}$

$x=2 \to y = \frac{2(2)-3}{2-1} = \frac{4-3}{1} = \frac{1}{1} = 1$

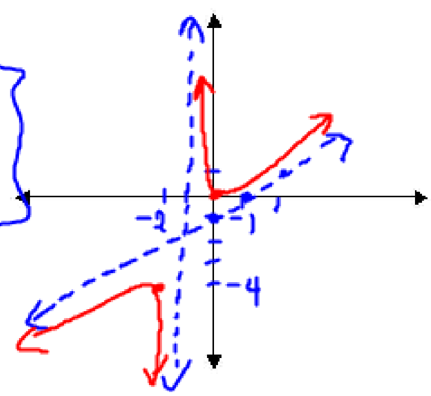
$\therefore (2,1)$

Example 3: $y = \frac{x^2}{x+1}$

y int $\to$ $x=0$

$y = \frac{0}{0+1} = 0$ $\therefore (0,0)$

$x=-1 \to$ V.A.



The degree of the expression in the numerator is greater than the degree of the expression in the denominator.

Using synthetic division to divide,

$$\begin{array}{r|rrr} -1 & 1 & 0 & 0 \\ & \downarrow & -1 & 1 \\ \hline & 1 & -1 & 1 \end{array}$$

$1x-1=y$

Oblique Asymptote

$x=-2$
$y = \frac{(-2)^2}{-2+1}$
$= \frac{4}{-1} = -4$

$\therefore (-2,-4)$

The quotient is $x - 1$ and the remainder is 1. The remainder can be written as $y = x - 1 + \frac{1}{x+1}$.

So, as $|x| \to \infty, \frac{1}{x+1} \to 0$, so $y \to x - 1 + 0$, or $x - 1$, and the graph has an oblique asymptote with the equation $y = x - 1$.

Determining Horizontal Asymptotes:

The graph of a rational function $y = \frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ y have no common factors, has:

- A horizontal asymptote at $y = 0$ if the degree of $f(x)$ is less than the degree of $g(x)$.
- A horizontal asymptote at $y = \frac{a}{b}$, where a is the leading coefficient of $f(x)$, and b is the lead coefficient of $g(x)$ if the degrees of $f(x)$ and $g(x)$ are equal.
- An oblique asymptote if the degree of $f(x)$ is 1 more than the degree of $g(x)$.

Use the equation of each function to predict whether its graph has horizontal or oblique asymptotes. Write the equations of these asymptotes.

Example 4: $y = \frac{x^2+6x-7}{x+2}$ → Degree of num. > Degree of denom.

∴ Oblique Asymptote

→ $y = x + 4$

$$\begin{array}{r|rrr} -2 & 1 & 6 & -7 \\ & \downarrow & -2 & -8 \\ \hline & 1 & 4 & \boxed{-15} \end{array}$$

Example 5: $y = \frac{x+2}{x^2+6x+9}$ Degree of num. < degree of denom.

∴ H.A. @ $y=0$

For each graph below:

- Determine any non-permissible values of x, and whether each indicates a hole or a vertical asymptote.
- Determine the equations of any horizontal or oblique asymptotes.
- Determine the domain.

Example 6: $y = \frac{(x^2+x-12)}{x+3}$ $= \frac{(x+4)(x-3)}{(x+3)}$

Domain

$x \neq -3$

$\{x \mid x \neq -3, x \in \mathbb{R}\}$

$(-\infty, -3) \cup (-3, \infty)$

$x = -3 \rightarrow$ V.A.

Oblique Asymptote ∴ use synthetic division

$$\begin{array}{r|rrr} -3 & 1 & 1 & -12 \\ & & -3 & 6 \\ \hline & 1 & -2 & -6 \end{array}$$

↳ at $y = x - 2$

$y = \frac{x^2-2x-3}{-x^2+1}$

Example 7: $y = \frac{x^2-2x-3}{1-x^2}$ $= \frac{(x-3)(x+1)}{-(x^2-1)} = \frac{(x-3)(x+1)}{-(x-1)(x+1)}$

$x \neq 1, -1$

Domain: $\{x \mid x \neq 1, x \neq -1, x \in \mathbb{R}\}$

H.A. @ $\frac{1}{-1} = -1$

@ $y = -1$

$x = 1 \rightarrow$ V.A.

$x = -1 \rightarrow$ Hole

HWK pg 451

1, 4 (no graph), 5, 6

2.4 Sketching Graphs of Rational Functions

To sketch the graph of a rational function:

- Determine holes or vertical asymptotes using the non-permissible values of x.
- Determine the equation of the horizontal or oblique asymptotes.
- Establish the behaviour of the graph near the asymptotes.
- Find the location of the intercepts.

Sketch the graph of these rational functions. Then state the domain and range.

Example 1: $y = \frac{x^2-5x+4}{1-x}$ $= \frac{(x-4)(x-1)}{-(x-1)}$

$\therefore$ Hole @ $x=1$

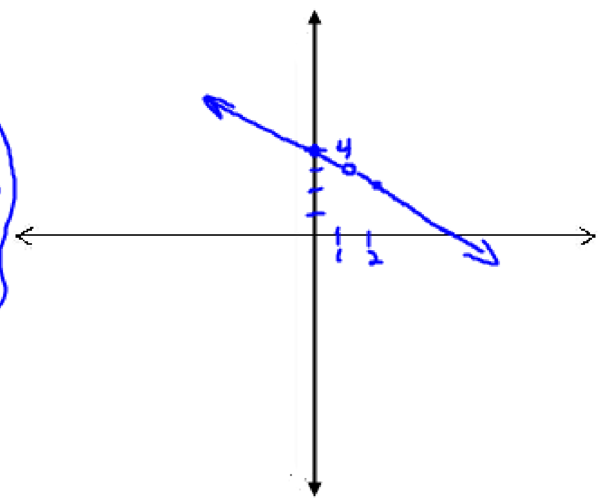
$y = \frac{x-4}{-1} = -x+4$

$\downarrow$

y-int @ 4

slope of $\frac{-1}{1}$

$\boxed{\begin{array}{l}\text{Domain} \\ \{x \mid x \neq 1, x \in \mathbb{R}\} \\ \text{Range} \\ \{y \mid y \neq 3, y \in \mathbb{R}\}\end{array}}$



Example 2: $y = -\frac{2x^2}{x^2-25} = \frac{-2x^2}{(x-5)(x+5)}$

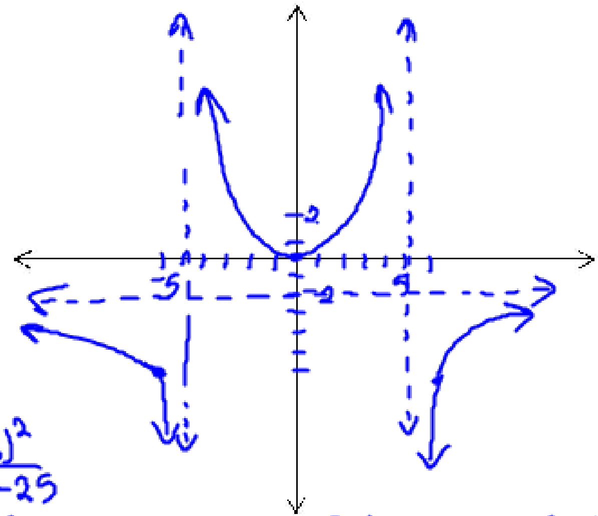
$x \neq 5, -5$

V.A. @ $x = 5$
$x = -5$

H.A. → degree of num. & denom = 2

∴ H.A. @ $\frac{-2}{1}$

$y = -2$



$x = 0$
$y = \frac{-0}{-25}$
$y = 0$
∴ $(0,0)$

$x = -6$
$y = \frac{-2(-6)^2}{(-6)^2-25}$
$= \frac{-2(36)}{11}$
$= \frac{-72}{11} \approx -6.5$

$x = 6$
$y = \frac{-2(6)^2}{(6)^2-25}$
$y = \frac{-72}{11} \approx -6.5$

D: $\{x \mid x \neq 5, x \neq -5, x \in \mathbb{R}\}$
R: $\{y \mid y \neq -2, y \in \mathbb{R}\}$
or → $\{y \mid y < -2, y > 0, y \in \mathbb{R}\}$

Example 3: $y = \frac{-2x^2+5x-2}{x-1} = \frac{-(2x^2-5x+2)}{(x-1)} = \frac{-(2x-1)(x-2)}{(x-1)}$

VA @ $x = 1$

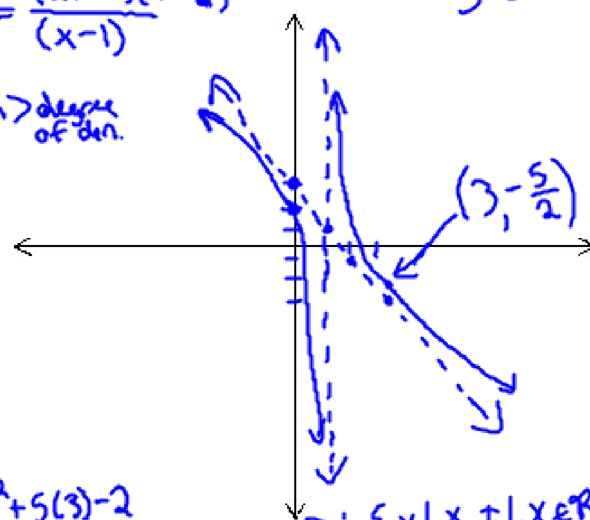
Oblique Asym → degree of num > degree of den.

$$\begin{array}{r|rrr} 1 & -2 & 5 & -2 \\ & \downarrow & -2 & 3 \\ \hline & -2 & 3 & \boxed{1} \end{array}$$

$y = -2x + 3$

yint → $x = 0$
$y = \frac{-2}{-1} = 2$

$x = 3$
$y = \frac{-2(3)^2+5(3)-2}{3-1}$
$= \frac{-18+15-2}{2}$
$= -\frac{5}{2}$



D: $\{x \mid x \neq 1, x \in \mathbb{R}\}$
R: $\{y \mid y \in \mathbb{R}\}$

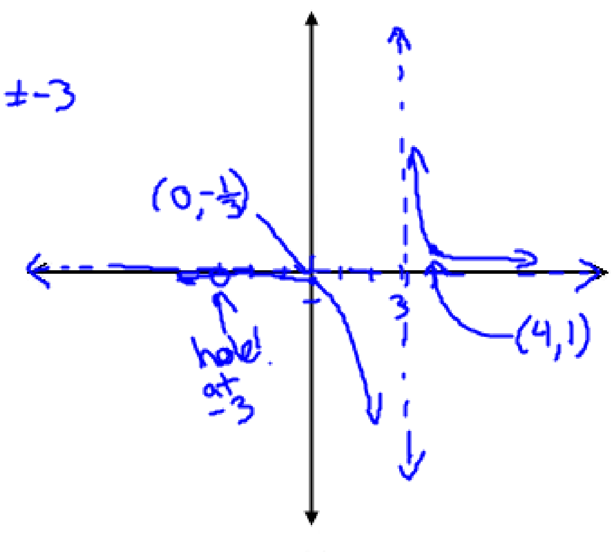
Example 4: $y = \frac{x+3}{x^2-9}$

$y = \frac{(x+3)}{(x-3)(x+3)} = \frac{1}{x-3}, x \neq -3$

Hole @ -3

V.A. @ 3

H.A. @ y=0



y int

$x=0$

$y = \frac{1}{0-3} = \frac{1}{-3}$

$\therefore (0, -\frac{1}{3})$

$x = 4$

$y = \frac{1}{4-3} = \frac{1}{1}$

$\therefore (4,1)$

2.5 Solving Radical and Rational Equations

Recall: The solution of an equation is the zero of the related function; that is, the value of x for which $f(x) = 0$.

Example 1: Solve the following equations algebraically.

a. $\sqrt{3x-1} + 1 = 2$

$-1 \quad -1$

Get the square root by itself on one side of the equation

$(\sqrt{3x-1})^2 = (1)^2$

$3x - 1 = 1$

$+1 \quad +1$

$\frac{3x}{3} = \frac{2}{3}$

$\boxed{x = \frac{2}{3}}$

Check:

$\sqrt{3(\frac{2}{3})-1} + 1$

$\sqrt{2-1} + 1$

$\sqrt{1} + 1$

$1 + 1 = 2$ ✓

b. $(\sqrt{2x+3})^2 = (\sqrt{4x-1})^2$

$2x + 3 = 4x - 1$

$-4x \quad -4x$

$-2x + 3 = -1$

$-3 \quad -3$

$\frac{-2x}{-2} = \frac{-4}{-2}$

$x = 2$

Check:

LHS	RHS
$\sqrt{2(2)+3}$	$\sqrt{4(2)-1}$
$\sqrt{4+3}$	$\sqrt{8-1}$
$\sqrt{7}$ ✓	$\sqrt{7}$ ✓

LCD: $(x-1)(x+4)$

c. $\frac{3}{x-1} + 2 = \frac{2}{x+4}$

$$\frac{3(x+4)}{(x-1)(x+4)} + \frac{2(x-1)(x+4)}{(x-1)(x+4)} = \frac{2(x-1)}{(x-1)(x+4)}$$

*Get rid of denominators

$$3(x+4) + 2(x-1)(x+4) = 2(x-1)$$
$$3x + 12 + 2[x^2 + 3x - 4] = 2x - 2$$
$$3x + 12 + 2x^2 + 6x - 8 = 2x - 2$$
$$3x + 12 + 2x^2 + 6x - 8 - 2x + 2 = 0$$
$$2x^2 + 7x + 6 = 0$$
$$(2x+3)(x+2) = 0 \qquad \therefore x = -\tfrac{3}{2}, -2$$

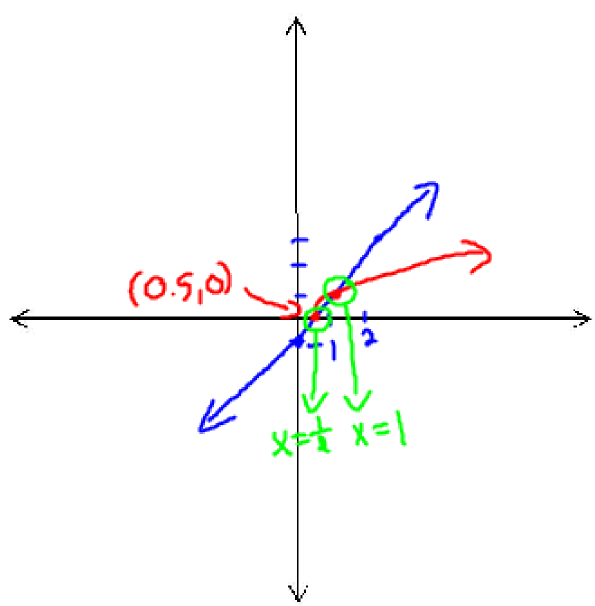
Example 2: Solve the following equations graphically.

a. $\sqrt{2x-1} = 2x - 1$

Graph $y = \sqrt{2x-1}$ (red)
$y = 2x - 1$ (blue)

And see where they intersect.

$\therefore$ The solutions are $x = \frac{1}{2}$ and 1



b. $\frac{x^2-10x+25}{x-5}=\frac{x+1}{x^2-4x-5}$

$y=\frac{(x-5)(x-5)}{x-5}$

$y=x-5$

Hole at $x=5$

$y=\frac{x+1}{(x-5)(x+1)}$

$y=\frac{1}{x-5}$

Hole at $x=-1$

V.A. @ $x=5$

H.A. @ $y=0$

$\underline{x=6}$

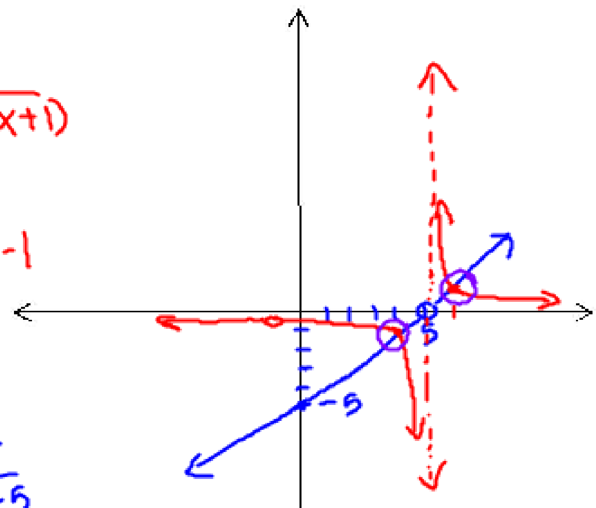
$y=\frac{1}{6-5}$

$y=1$

$\underline{x=4}$

$y=\frac{1}{4-5}$

$y=-1$



$\boxed{\therefore x=4,6}$