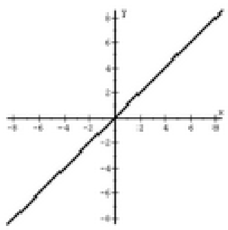
Unit 3: Transformations

3.0 Parent Functions

You need to be familiar with the following "parent functions".

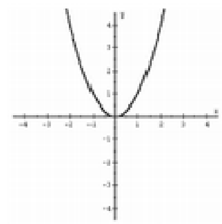
Linear Function



$f(x) = x$

(-1, -1)
(0, 0)
(1, 1)

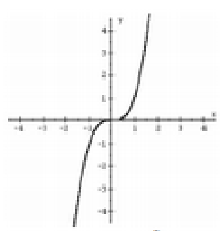
Quadratic Function



$f(x) = x^2$

(-1, 1)
(0, 0)
(1, 1)

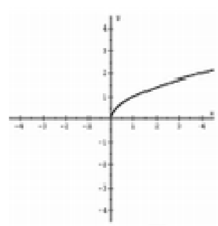
Cubic Function



$f(x) = x^3$

(-1, -1)
(0, 0)
(1, 1)

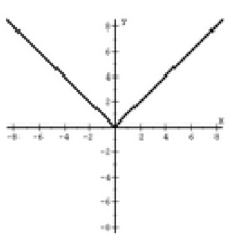
Square Root Function



$f(x) = \sqrt{x}$

(0, 0)
(1, 1)
(4, 2)

Absolute Value Function

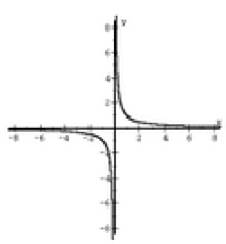


$f(x) = |x|$

(-1, 1)
(0, 0)
(1, 1)

Rational Function

$y \neq 0$
$x \neq 0$



$f(x) = \frac{1}{x}$

(-1, -1)
(1, 1)

3.1 Translations

The graph of $\boldsymbol{y - k = f(x - h)}$ is the image of the graph of $y = f(x)$ after a vertical translation of k units, and a horizontal translation of h units.

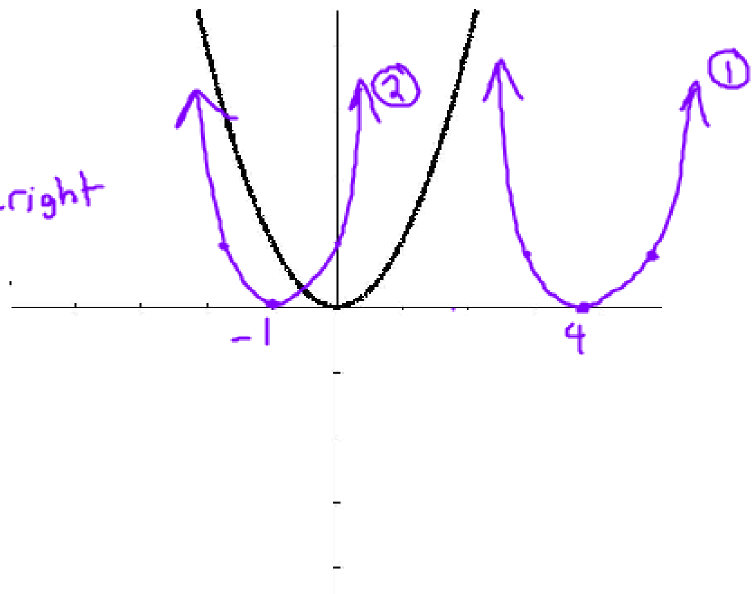
Example 1: $y = x^2$

(*i*) $y = (x-4)^2$

Move 4 to the right

(*ii*) $y = (x+1)^2$

Move 1 unit to the left.



Example 2 : $y = |x|$

(*i*) $y + 1 = |x|$

$y = |x| - 1$

Move 1 down.

(*ii*) $y - 3 = |x|$

$y = |x| + 3$

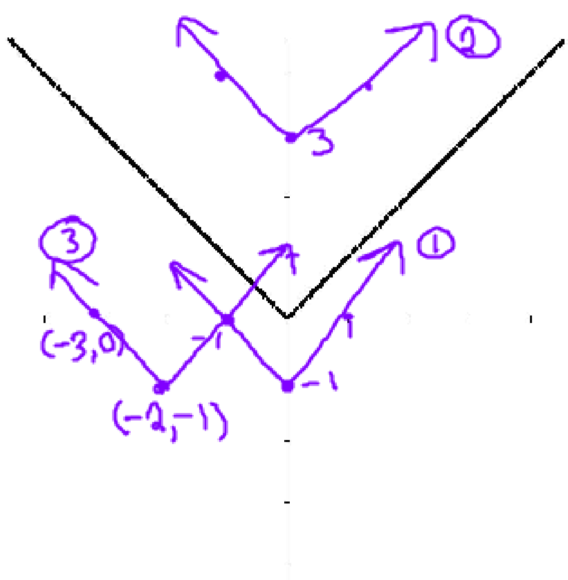
Move 3 up.

(*iii*) $y + 1 = |x + 2|$

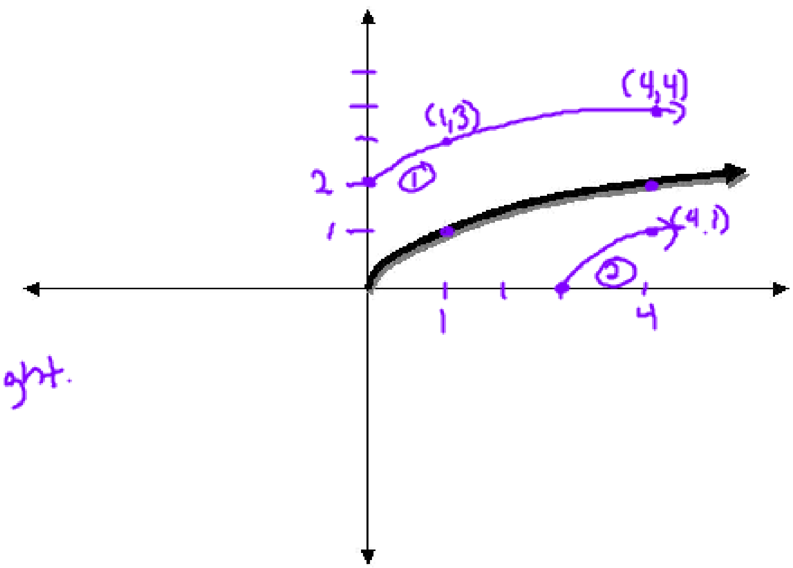
$y = |x+2| - 1$

Move down 1.

Move left 2.



Example 3: $y = \sqrt{x}$



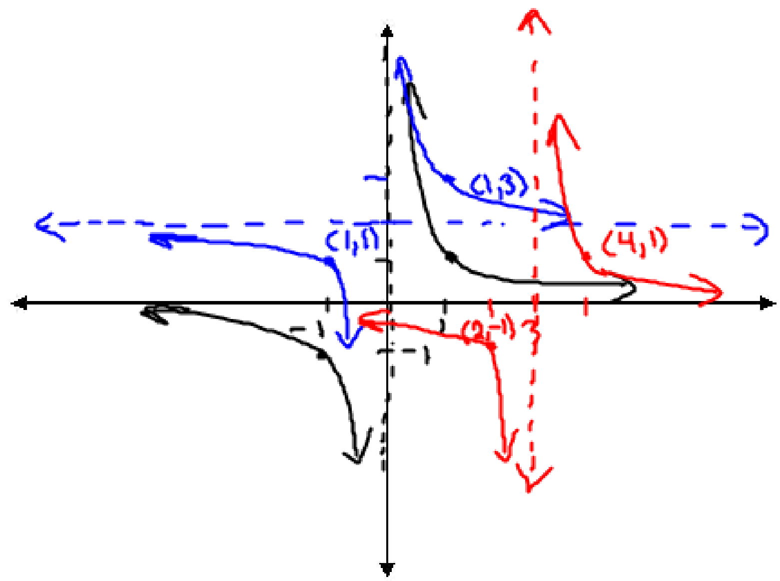
(i) $\quad y = \sqrt{x} + 2$

Moves 2 up.

(ii) $\quad y = \sqrt{x-3}$

Moves 3 to the right.

Example 4: $y = \frac{1}{x}$ Black $= \frac{1}{X}$



(i) $\quad y = \frac{1}{x} + 2$

Moves up 2.

(ii) $\quad y = \frac{1}{x-3}$

Move 3 to the right.

The graph of $\boldsymbol{y} = \boldsymbol{f}\,(\boldsymbol{x} - \boldsymbol{h})$ is a horizontal translation of the graph of $\boldsymbol{y} = \boldsymbol{f}\,(\boldsymbol{x})$. When $h > 0$, the graph is translated h units right. When $h < 0$, the graph is translated $|h|$ units left.

The graph of $\boldsymbol{y} - \boldsymbol{k} = \boldsymbol{f}\,(\boldsymbol{x})$ is a vertical translation of the graph of $\boldsymbol{y} = \boldsymbol{f}\,(\boldsymbol{x})$. When $k > 0$, the graph is translated k units up. When $k < 0$, the graph is translated $|k|$ units down.

Example 5: The graph of $y = \frac{1}{x}$ is translated 3 units left and 2 units up. What is the equation of the image graph?

↳ after transformations

$$y = \frac{1}{x+3} + 2$$

↑ same as what you expect

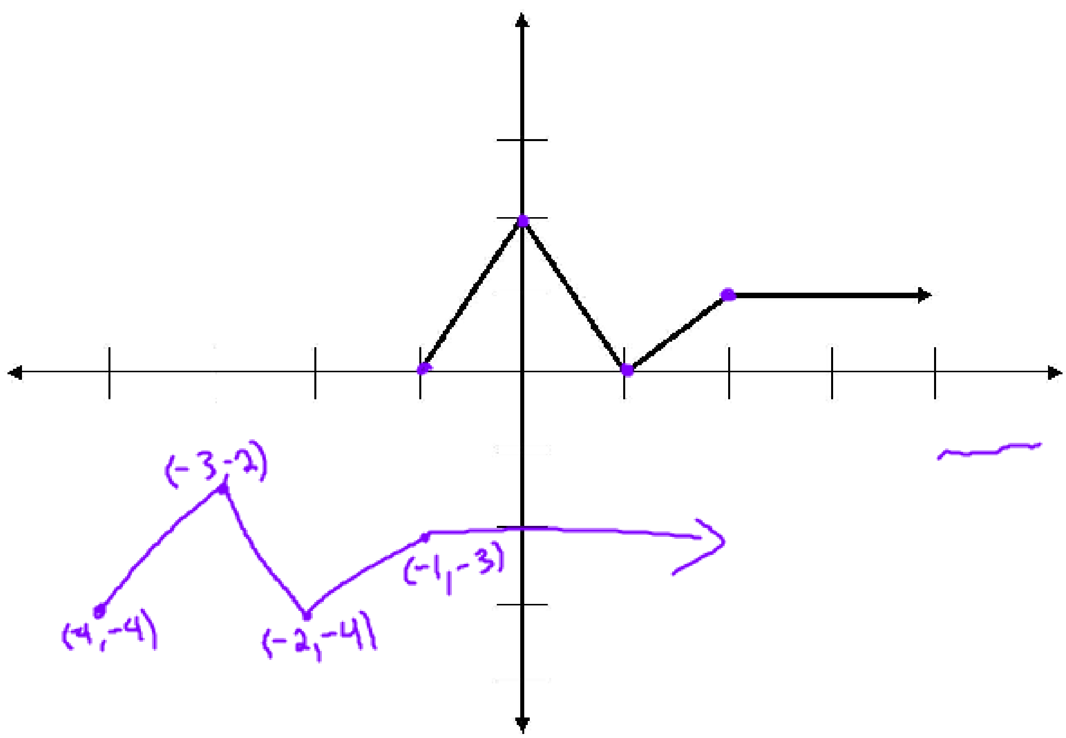
↑ opposite of what you think

Example 6: Given the graph of $y = f(x)$ below, sketch the graphs of:

(i) $f(x+3)$ Move 3 left.

(ii) $f(x) - 4$ Move 4 down.

(iii) $f(x+3) - 4$ Move 3 left & 4 down.



Example 7: Describe how the graph of $y = \frac{1}{x^2}$ could have been translated to create the graph of each function below. What are the equations of the asymptotes of each image graph?

a. $y - 3 = \frac{1}{x^2}$

$y = \frac{1}{x^2} + 3$

Move the graph of $y = \frac{1}{x^2}$ up 3.

b. $y + 4 = \frac{1}{(x+3)^2}$

$y = \frac{1}{(x+3)^2} - 4$

Move the graph of $y = \frac{1}{x^2}$ down 4 and left 3.

3.2 Transformations – Reflections

A graph may be reflected in the x-axis or the y-axis. For a function $y = f(x)$

- The graph of $y = -f(x)$ is the image of the graph of $y = f(x)$ after a reflection in the x-axis. A point (x, y) on $y = f(x)$ corresponds to the point $(x, -y)$ on $y = -f(x)$.
- The graph of $y = f(-x)$ is the image of the graph of $y = f(x)$ after a reflection in the y-axis. A point (x, y) on $y = f(x)$ corresponds to the point $(-x, y)$ on $y = f(-x)$.

Example 1:

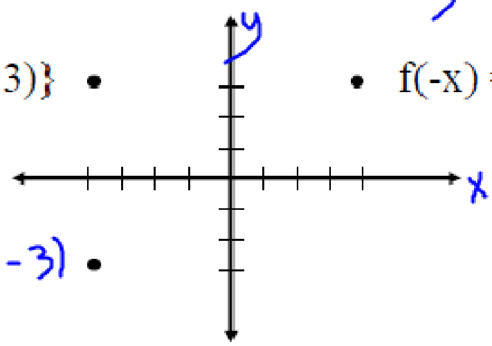
If $y = f(x)$, then $y = -f(x)$ is obtained by reflect through the x-axis

If $y = f(x)$, then $y = f(-x)$ is obtained by flipping through the y-axis

f(x) = {(-4, 3)}

f(-x) = (4, 3)

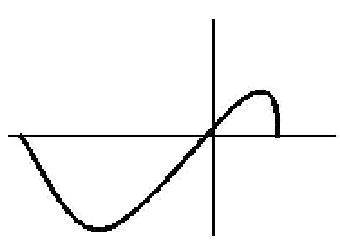
-f(x) = (-4, -3)

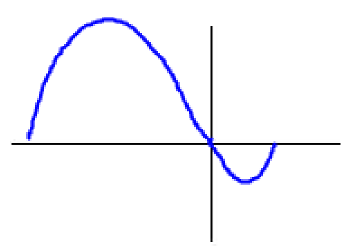


Example 2:

Reflecting over the x-axis

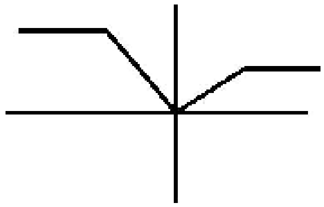
$y = f(x)$ $\qquad\qquad$ $y = -f(x)$

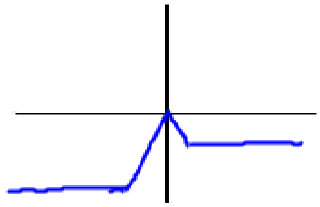




Example 3 :

$y = f(x)$ $\qquad\qquad$ $y = -f(x)$

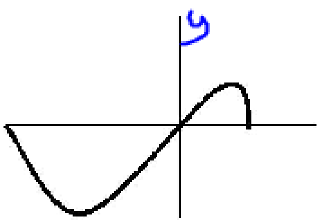


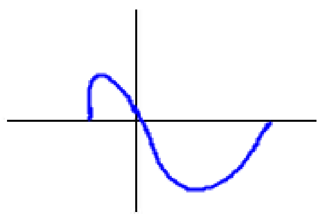


Example 4:

Reflecting over the *y*-axis

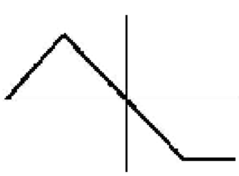
$y = f(x)$ $\qquad\qquad$ $y = f(-x)$

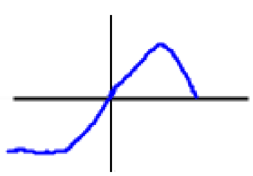




Example 5 :

$y = f(x)$ $\qquad\qquad$ $y = f(-x)$



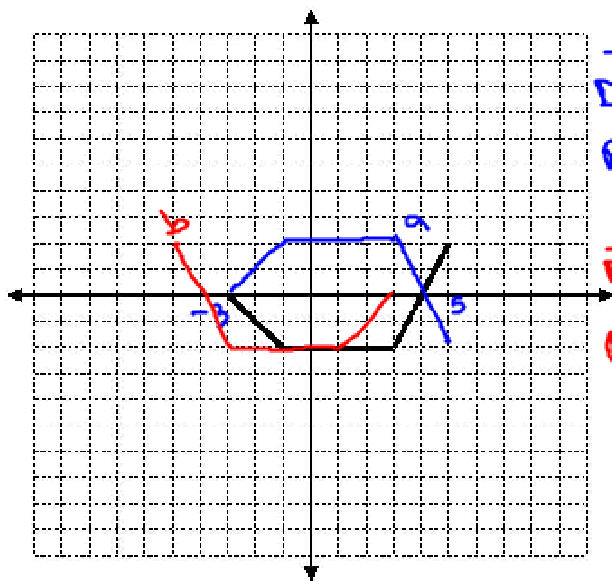


Example 6: The graph of $y = g(x)$ is given.

flip through x
flip through y

a. Sketch the graph of $y = -g(x)$. State the domain and range of each function.

b. Sketch the graph of $y = g(-x)$. State the domain and range of each function.



a

D: $[-3,5]$

R: $[-2,2]$

b

D: $[-5,3]$ $\{x \mid -5 < x \leq 3\}$

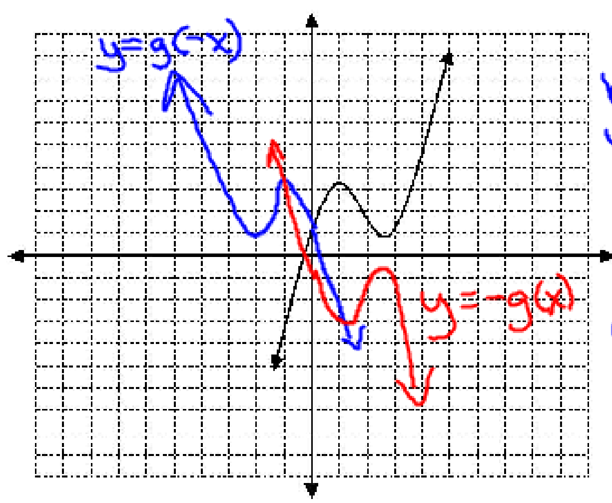
R: $[-2,2]$

$\{y \mid -2 \leq y \leq 2\}$

Example 7: The graph of $y = g(x)$ is given.

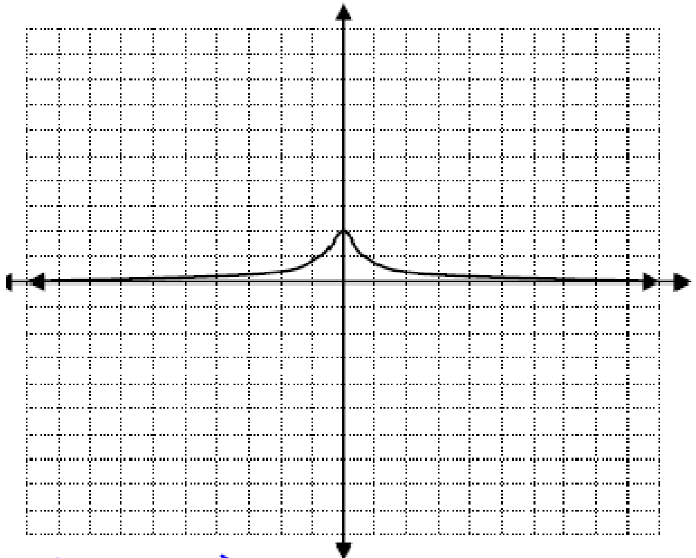
a. Sketch the graph of $y = -g(x)$. State the domain and range of each function.

b. Sketch the graph of $y = g(-x)$. State the domain and range of each function.



D & R for both functions $y=-g(x)$ & $y=g(-x)$ is $(-\infty,\infty)$

Example 8: The graph of $y = \frac{1}{-2.5x^2 - 0.5}$ was reflected in the x-axis and its image is shown. What is the equation of the image?



$$y = -\left(\frac{1}{-2.5x^2 - 0.5}\right) = \frac{-1}{-(2.5x^2 + 0.5)}$$

$$\therefore y = \frac{1}{2.5x^2 + 0.5}$$

3.3 Stretching and Compressing Graphs of Functions

Vertical Stretch or Compression: $y = a\,f(x)$

The graph of $y = af(x)$ is the image of the graph of $y = f(x)$ after a vertical stretch, compression or reflection. Point (x, y) on $y = f(x)$ corresponds to point (x, ay) on $y = af(x)$.

- When $0 < |a| < 1$, there is a vertical compression by a factor of $|a|$
- When $|a| > 1$, there is a vertical stretch by a factor of $|a|$
- When a < 0, there is a reflection in the x-axis as well as the stretch or compression.

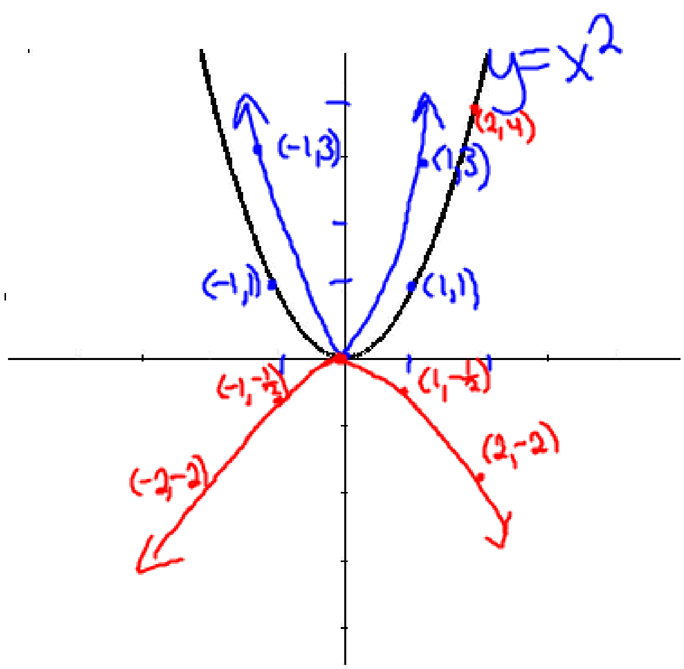
Ex. 1: $y = f(x) = x^2$

$y_1 = 3f(x)$

$y_2 = -\frac{1}{2}f(x)$

$(x, y) \rightarrow (x, 3y)$

$(x, y) \rightarrow (x, -\frac{1}{2}y)$



Ex. 2

$y = f(x)$

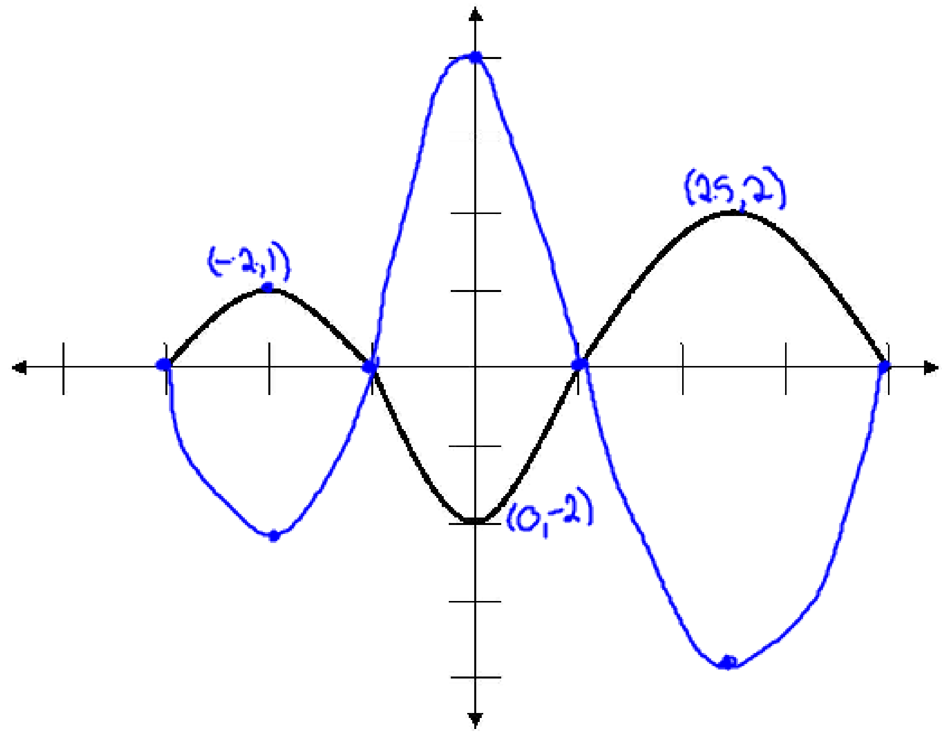
$y_1 = -2f(x)$

$(x, y) \Rightarrow (x, -2y)$

$(-2, 1) \Rightarrow (-2, -2)$

$(0, -2) \Rightarrow (0, 4)$

$(2.5, 2) \Rightarrow (2.5, -4)$



Ex. 3

$y = f(x)$

$y_1 = -\frac{1}{2}f(x)$

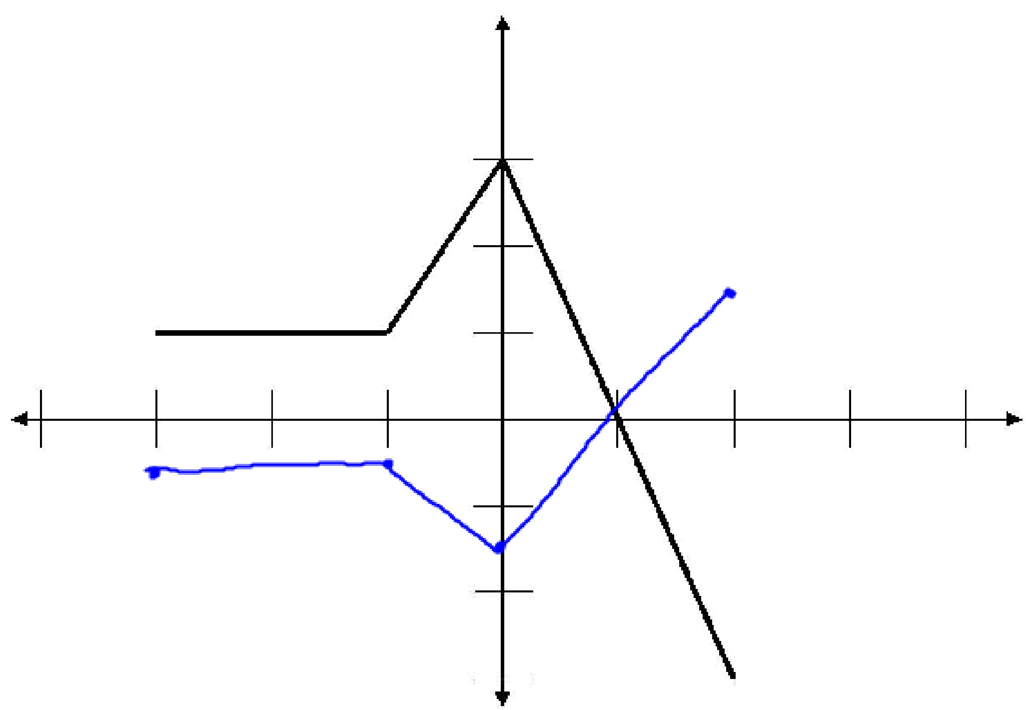
$(x, y) \Rightarrow (x, -\frac{1}{2}y)$

$(-3, 1) \Rightarrow (-3, -\frac{1}{2})$

$(-1, 1) \Rightarrow (-1, -\frac{1}{2})$

$(0, 3) \Rightarrow (0, -\frac{3}{2})$

$(2, -3) \Rightarrow (2, +\frac{3}{2})$



Horizontal Stretch or Compression $\quad y = f(bx)$

The graph of $y = f(bx)$ is the image of the graph of $y = f(x)$ after a horizontal stretch, compression or reflection. Point (x, y) on $y = f(x)$ corresponds to point $\left(\frac{x}{b}, y\right)$ on $y = f(bx)$.

- When $0 < |b| < 1$, there is a horizontal stretch by a factor of $\frac{1}{b}$
- When $|b| > 1$, there is a horizontal compression by a factor of $\frac{1}{b}$
- When b < 0, there is a reflection in the y-axis as well as the stretch or compression.

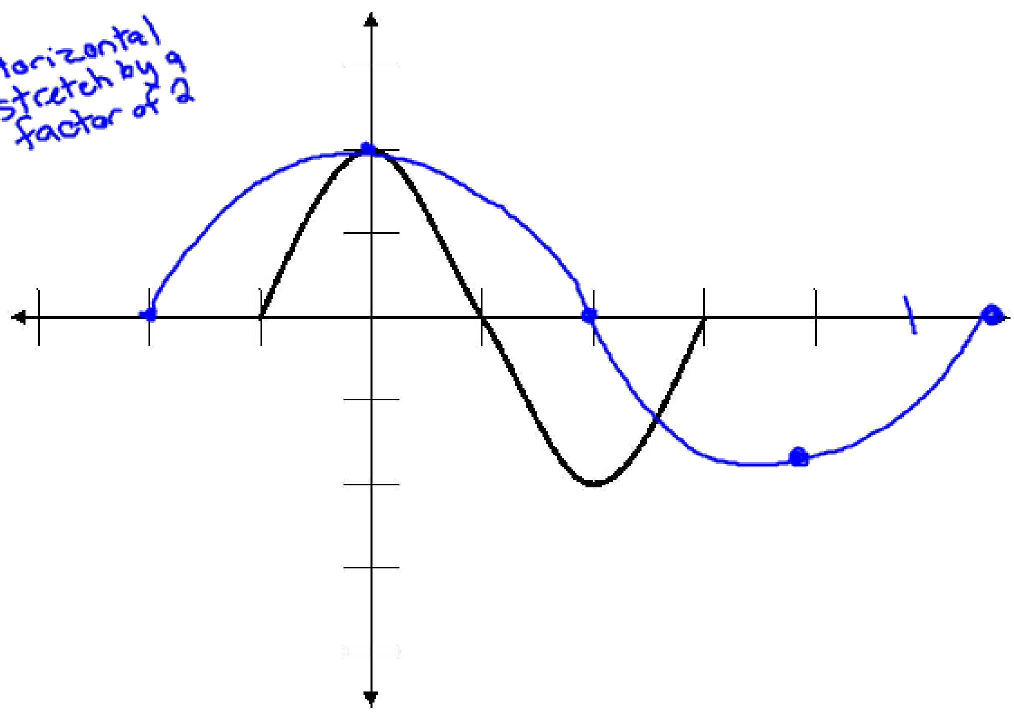
Ex. 4

$y = f(x)$

$y_1 = f\left(\frac{1}{2}x\right)$ ← Horizontal stretch by a factor of 2

$(x, y) \Rightarrow (2x, y)$

$(-1, 0) \rightarrow (-2, 0)$
$(0, 2) \rightarrow (0, 2)$
$(1, 0) \rightarrow (2, 0)$
$(2, -2) \rightarrow (4, -2)$
$(3, 0) \rightarrow (6, 0)$



Ex. 5

$y = f(x)$

$y_1 = f(2x)$

$(x, y) \Rightarrow (\tfrac{1}{2}x, y)$

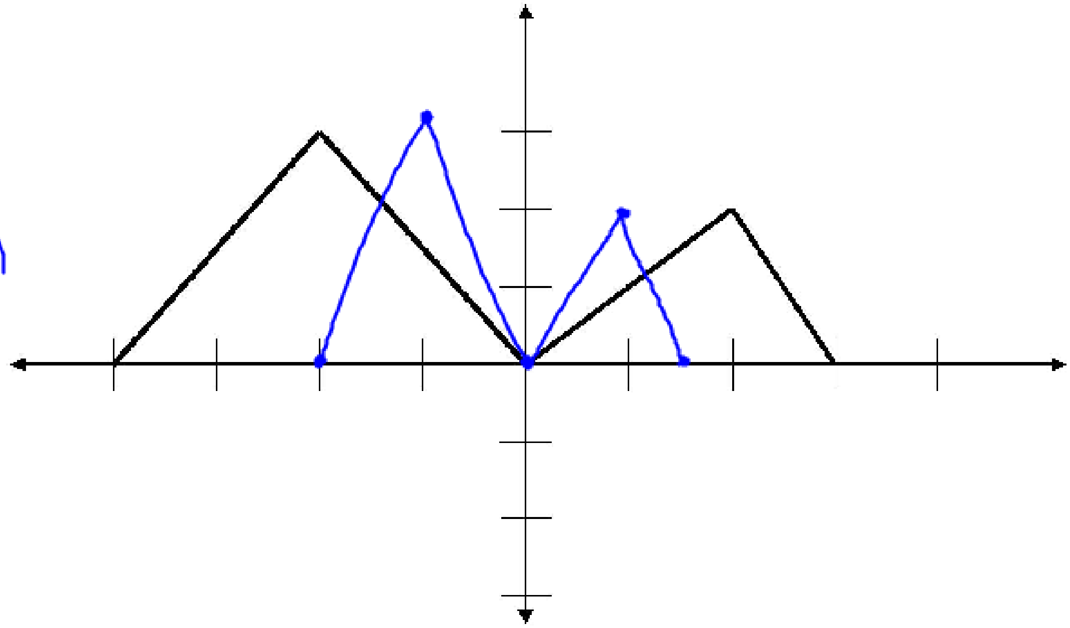
$(-4,0) \Rightarrow (-2,0)$

$(-2,3) \Rightarrow (-1,3)$

$(0,0) \Rightarrow (0,0)$

$(2,2) \Rightarrow (1,2)$

$(3,0) \Rightarrow (\tfrac{3}{2},0)$

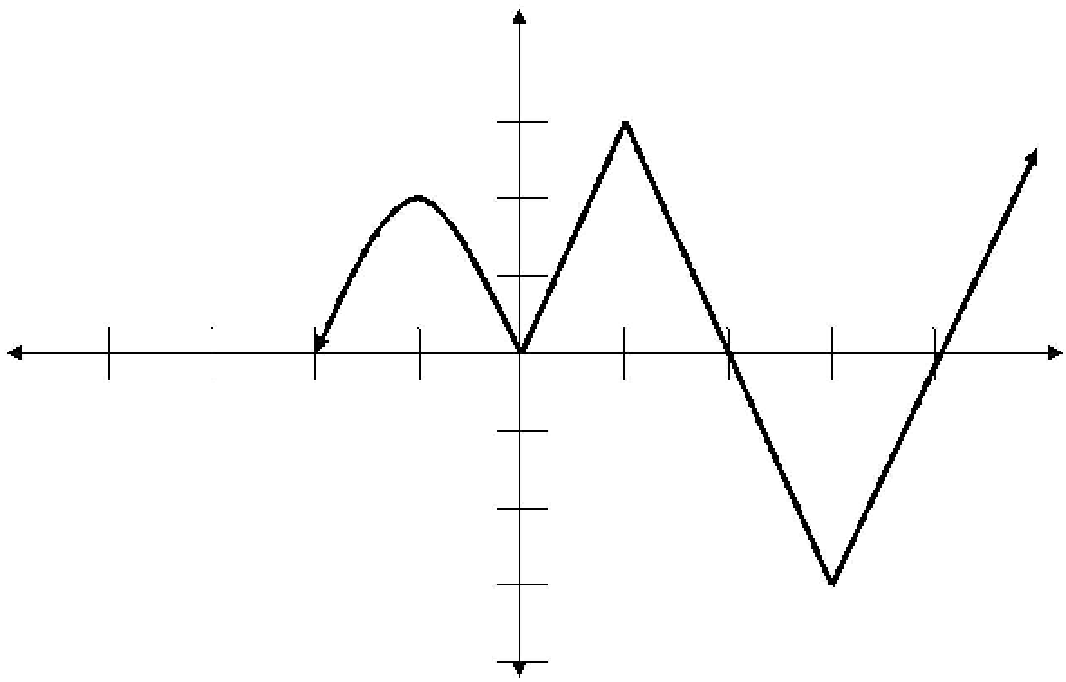


Ex. 6

Given the graph of $y = f(x)$, sketch

a. $y_1 = f(2x)$ → x-values halved

b. $y_2 = f\left(\frac{1}{2}x\right)$ → x-values doubled



Combining Horizontal and Vertical Movement

The point $(x. y)$ on $y = f(x)$ corresponds to the point $\left(\frac{x}{b}, ay\right)$ on $y = af(bx)$.

Ex. 7 Given the graph of $y = f(x)$, sketch $y = 2f(-0.5x)$

$-\frac{1}{2}$

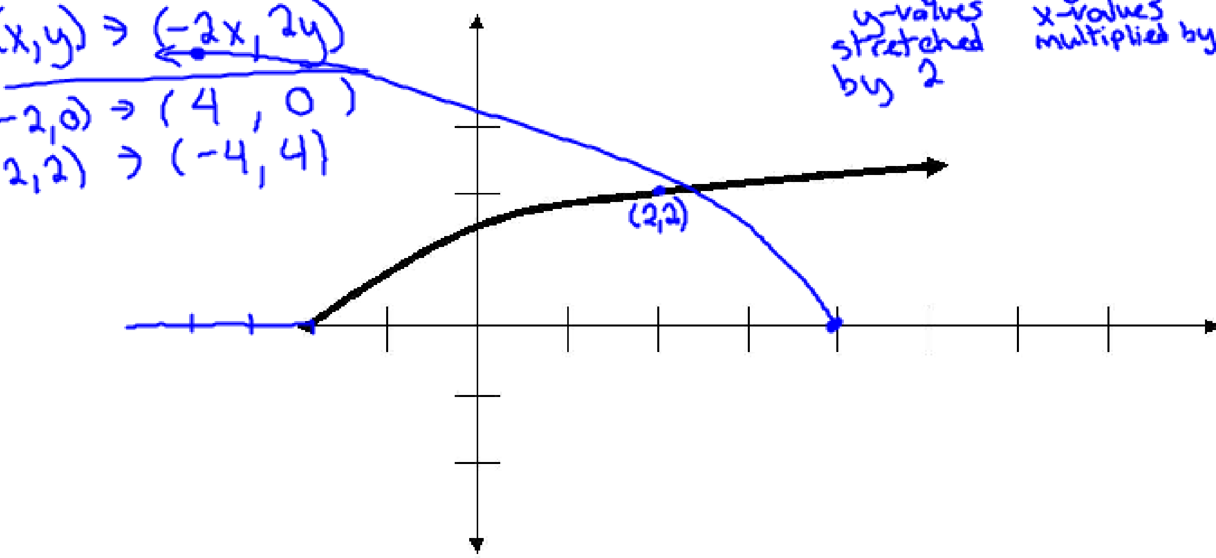
y-values stretched by 2

x-values multiplied by -2

$(x,y) \to (-2x, 2y)$

$(-2,0) \to (4, 0)$

$(2,2) \to (-4,4)$



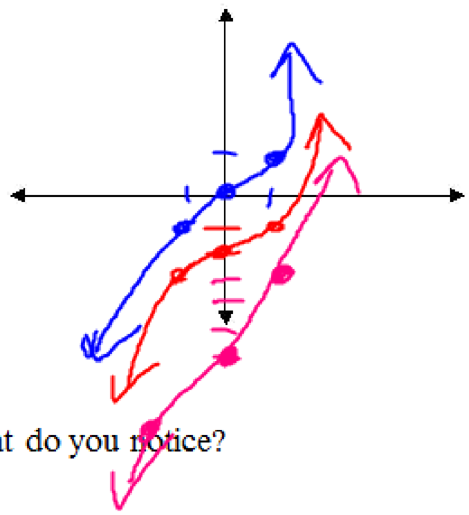
3.4 Combining Transformations of Functions

~~Practice: Using your graphing calculator.~~

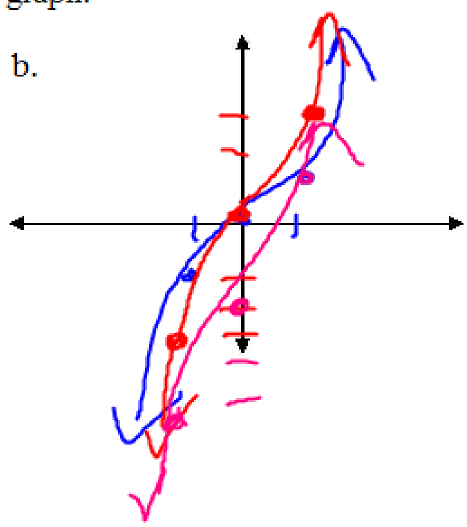
a. Graph $y = x^3$. Translate the graph 2 units down, and then stretch it vertically by a factor of 3. Sketch the graph.

b. Graph $y = x^3$. Stretch the graph by a factor of 3, and then translate the graph 2 units down. Sketch the graph.

a.



b.



What do you notice?

Combining Transformations

The graph of $y - k = af\big(b(x-h)\big)$ is the image of the graph of $y = f(x)$ after these transformations:

stretches & compressions & reflection

- A horizontal stretch or compression by a factor of $\frac{1}{|b|}$
- A reflection in the y-axis if $b < 0$;
- A vertical stretch or compression by a factor of $|a|$
- A reflection in the x-axis if $a < 0$.

Followed by:

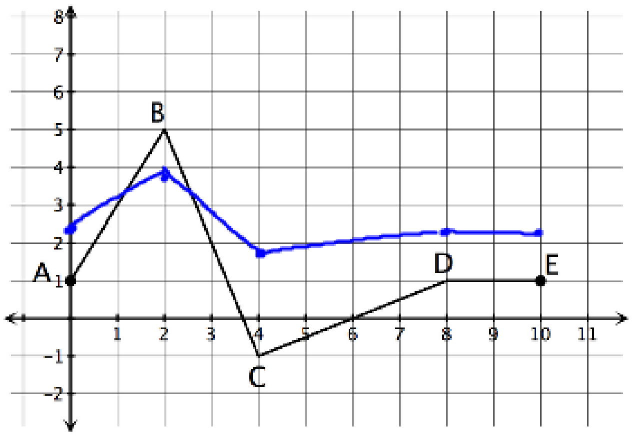
- A horizontal translation of h units
- A vertical translation of k units.

Point (x, y) on the graph of $y = f(x)$ corresponds to the point $\left(\frac{x}{b} + h, ay + k\right)$ on the graph of $y - k = af\big(b(x-h)\big)$; this is *the general transformation*.

Ex. 1. Here is the graph of $= f(x)$. Sketch the graph of its image after a vertical compression by a factor of $\frac{1}{3}$ then a translation of 2 units up.

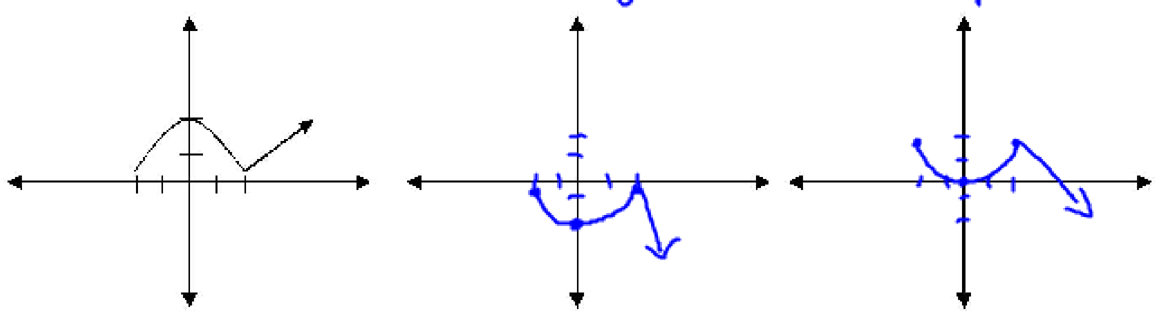
$(x, y) \rightarrow (x, \frac{1}{3}y+2)$

$(0,1) \rightarrow (0, \frac{7}{3})$
$(2,5) \rightarrow (2, \frac{11}{3})$
$(4,-1) \rightarrow (4, \frac{5}{3})$
$(8,1) \rightarrow (8, \frac{7}{3})$
$(10,1) \rightarrow (10, \frac{7}{3})$



Ex. 2. Use the graph of $y = g(x)$ given below to sketch $y = -g(x) + 2$.

-flip through x-axis   -move up 2

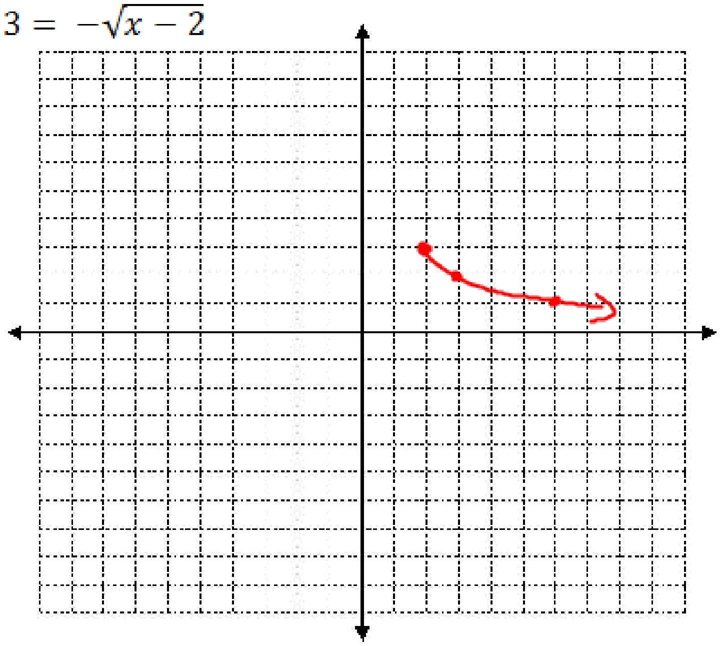


Ex. 3. Sketch the graph of $y - 3 = -\sqrt{x-2}$

$y = -\sqrt{x-2} + 3$
→ y values multiplied by −1
→ x values right 2
→ y values moved up 3

$(x, y) \rightarrow (x+2, -1y+3)$

$(0,0)$ $(2, 3)$
$(1,1)$ $(3, 2)$
$(4,2)$ $(6, 1)$



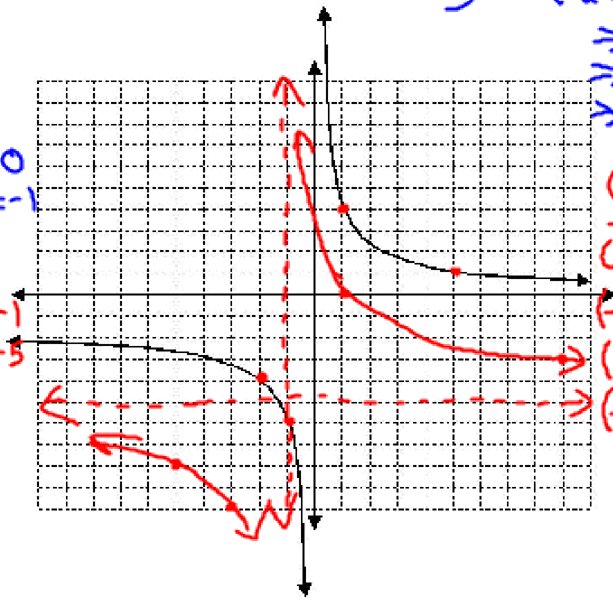
Ex. 4 Given the graph of $y = f(x)$, sketch the graph of $y + 4 = f\left(\frac{1}{2}(x+1)\right)$. $\frac{1}{2}x + \frac{1}{2}$

State the domain and range of each function. $y = f(\frac{1}{2}(x+1)) - 4$ $\frac{1}{2}(x+1)$

→ x values multiplied by 2
→ x values moved left 1
→ y values moved down 4

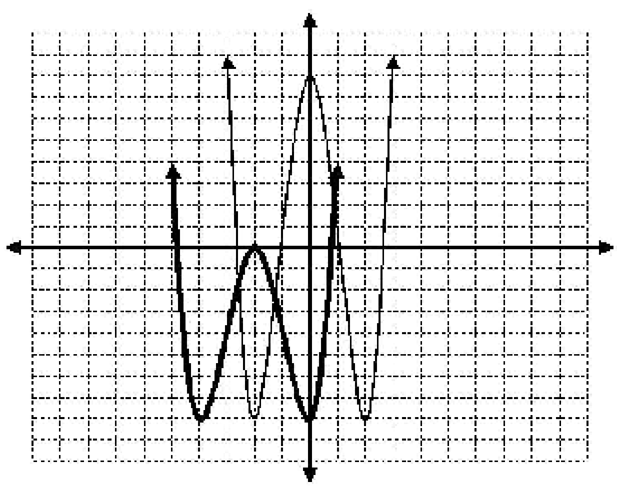
f(x)
V.A. @ x = 0
H.A. @ y = -1

g(x)
V.A. @ x = -1
H.A. @ y = -5



(x, y)	→	$(2x-1, y-4)$
$(1, 4)$	→	$(1, 0)$
$(-2, -4)$	→	$(-5, -8)$
$(5, 1)$	→	$(9, -3)$
$(-1, -6)$	→	$(-3, -10)$

Ex. 5. The graph of $y = g(x)$ is the image of the graph of $y = f(x)$ after a combination of transformations. Label corresponding points. Write and verify an equation for the image graph in terms of the function f.



3.5 Transformations – Inverse Relations

Recall: Solving for x.

a. $y = 3x - 4$
$+4 \quad +4$
$y+4=3x \qquad \frac{y+4}{3}=x$

b. $y = \frac{3x-5}{2}$
$2y=3x-5 \qquad 2y+5=3x \qquad \frac{2y+5}{3}=x$

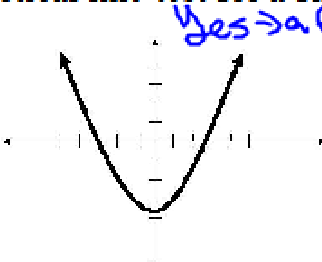
c. $y = 3x^2 - 5$
$+5 \quad +5$
$y+5=3x^2 \qquad \sqrt{\frac{y+5}{3}}=\sqrt{x^2} \qquad x=\pm\sqrt{\frac{y+5}{3}}$

d. $y = 2(x-3)^2 + 4$
$y-4=2(x-3)^2 \qquad \frac{y-4}{2}=(x-3)^2 \qquad \pm\sqrt{\frac{y-4}{2}}=x-3$
$3\pm\sqrt{\frac{y-4}{2}}=x$

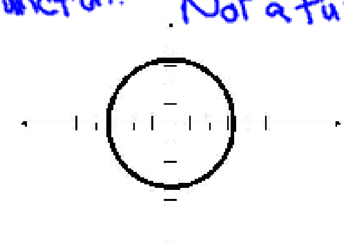
FUNCTION- a rule that assigns to each element in the domain exactly one value from the range.

Vertical line test for a function:

Yes ⇒ a function.

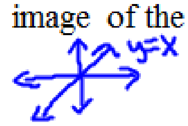


Not a function.



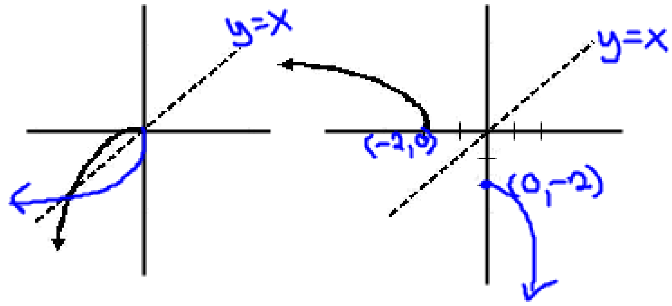
Reflecting in the line $y = x$.

- For a function $y = f(x)$, the graph of $x = f(y)$ is the image of the graph of $y = f(x)$ after a reflection in the line $y = x$.
- $y = f(x)$ and $x = f(y)$ are inverses of each other.
- A point (x, y) on $y = f(x)$ corresponds to the point (y, x) on $x = f(y)$.
- When the inverse is also a function, the notation $\boxed{f^{-1}(x)}$ is used to denote the inverse function. We say, "*f inverse of x*."



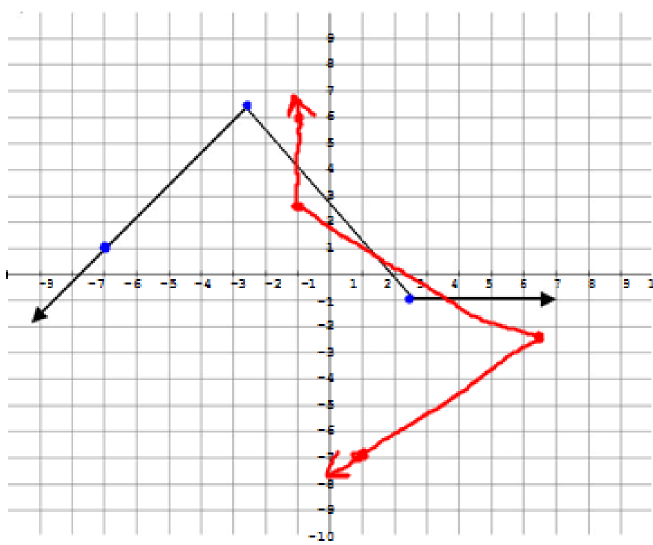
Ex. 1. Finding an Inverse – Given the graph

Reflect $y = f(x)$ over the line $y = x$ in both graphs below.



Ex. 2. Given the graph of the function $y = f(x)$.

a. Sketch the inverse of the given function:



$(x,y) \Rightarrow (y,x)$
$(-7,1) \Rightarrow (1,-7)$
$(-2.5,6.5) \Rightarrow (6.5,-2.5)$
$(2.5,-1) \Rightarrow (-1,2.5)$
$(6,-1) \Rightarrow (-1,6)$

b. Is the inverse a function? Why or why not?

Not a function $\Rightarrow$ 2 y-values for some values of x

c. State the domain and the range of the function and its inverse.

	$y=f(x)$	$y=f^{-1}(x)$
D:	$(-\infty,\infty)$	$(-\infty,6.5]$
R:	$(-\infty,6.5]$	$(-\infty,\infty)$

Note: The domain of $y = f(x)$ is the range of $x = f(y)$, and the range of $y = f(x)$ is the domain of $x = f(y)$.

To determine the equation of an inverse function, interchange x and y in the equation of the function, then solve the resulting equation for y.

Ex. 3.

a. Determine the equation of the inverse of $y = -x^2 + 4$.

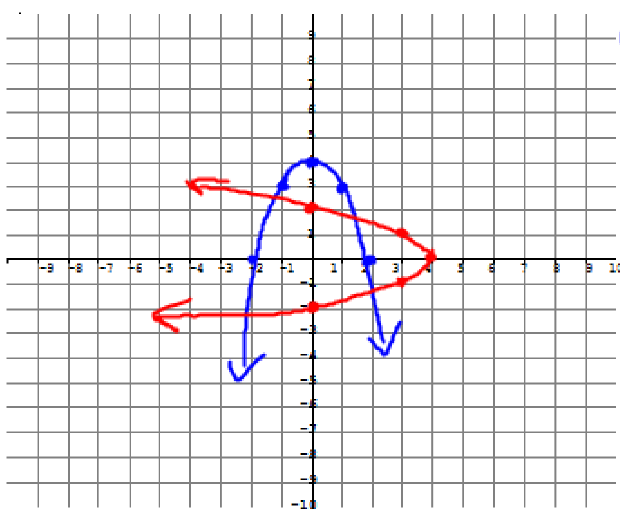
$x = -y^2 + 4$

$x - 4 = -y^2$

$-x + 4 = y^2$

$\therefore y = \pm\sqrt{-x+4}$

b. Sketch the graphs of $y = -x^2 + 4$ and its inverse.



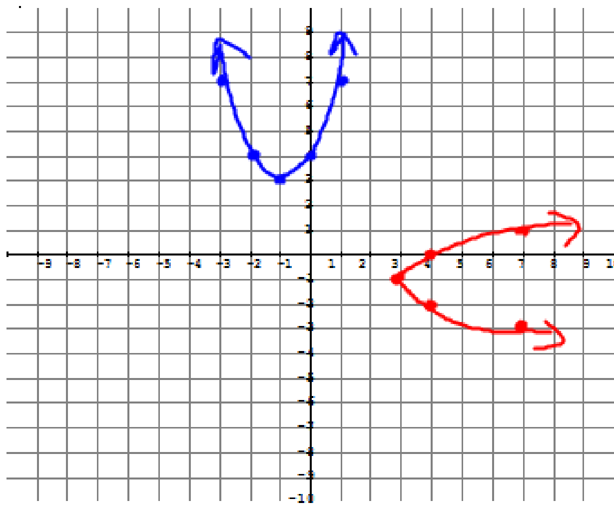
(x,y)	$\Rightarrow$	(y,x)
$(-2,0)$		$(0,-2)$
$(-1,3)$		$(3,-1)$
$(0,4)$		$(4,0)$
$(1,3)$		$(3,1)$
$(2,0)$		$(0,2)$

c. Is the inverse a function?

No.

Ex.4.

a. Sketch $y = (x + 1)^2 + 3$ and its inverse. V: $(-1, 3)$



(x, y)	$\Rightarrow (y, x)$
$(-3, 7)$	$(7, -3)$
$(-2, 4)$	$(4, -2)$
$(-1, 3)$	$(3, -1)$
$(0, 4)$	$(4, 0)$
$(1, 7)$	$(7, 1)$

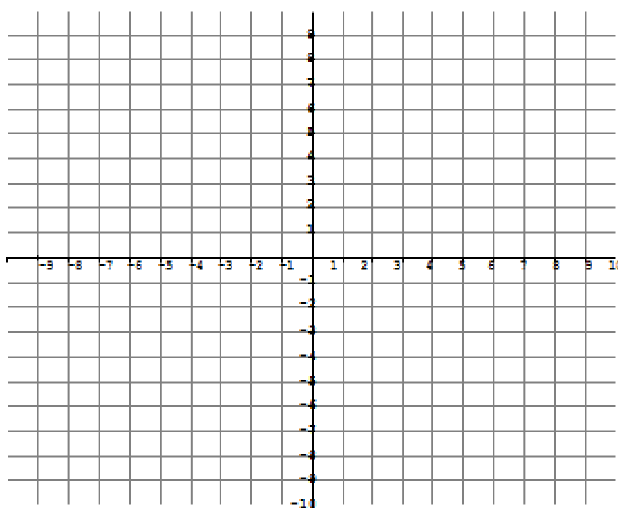
$x = (y+1)^2 + 3$

$x - 3 = (y+1)^2$

$\pm\sqrt{x-3} = y + 1$

b. Determine two ways to restrict the domain of $y = (x + 1)^2 + 3$ so that its inverse is a function. Write the equation of the inverse each time. Use a graph to illustrate each way.

$\pm\sqrt{x-3} - 1 = y$



Restrict Domain of $y = f(x)$ to:

$(-\infty, -1]$

$y = +\sqrt{x-3} - 1$

or

$[-1, \infty)$

$y = -\sqrt{x-3} - 1$

Ex. 5. Determine algebraically whether the functions are inverses of each other.

a. $y = 3x - 6$ and $y = \frac{(x-6)}{3}$ $\therefore$ Not inverses

Inverse of $y=3x-6$:

$x = 3y - 6$

$x + 6 = 3y$

$\frac{x+6}{3} = y$

b. $y = -x^2 + 3, x \geq 0$ and $y = \sqrt{3 - x}$

$x = -y^2 + 3$

$x - 3 = -y^2$

$-x + 3 = y^2$

$\sqrt{-x+3} = y$

$\sqrt{3-x} = y$ $\therefore$ They are inverses.