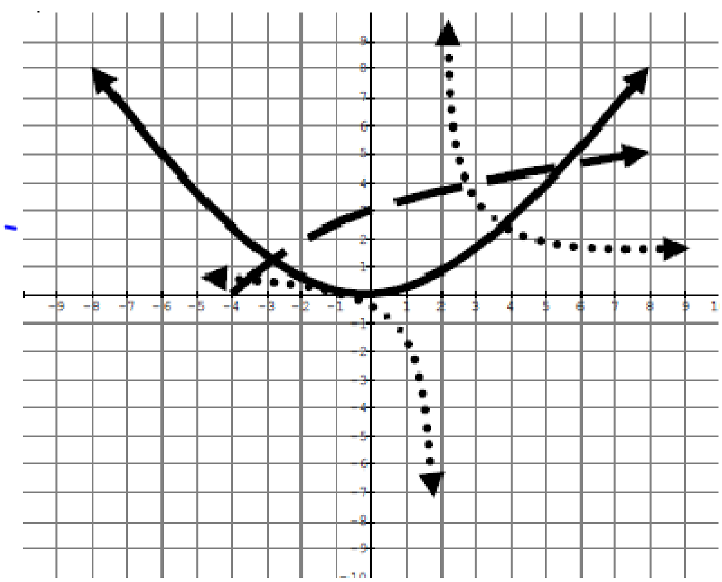
Unit 4: Combining Functions

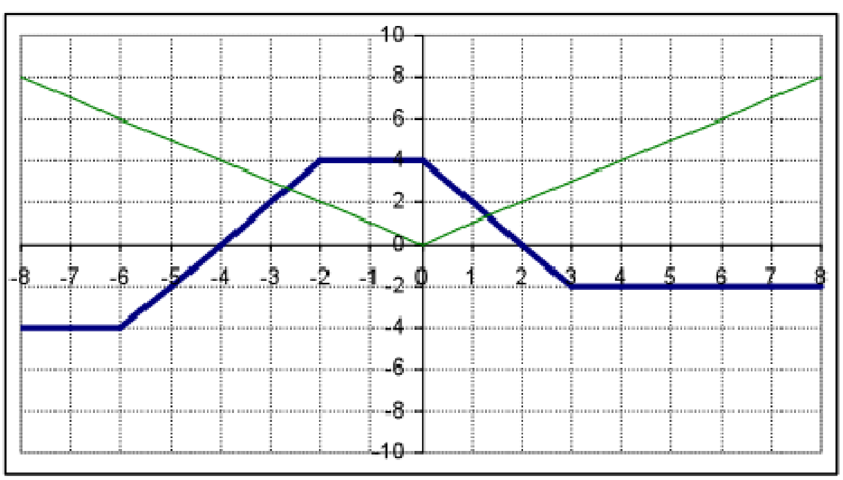
4.1 Combining Functions Graphically

Recall: Domain and Range



	D	R
——	$(-\infty,\infty)$	$[0,\infty)$
— —	$[-4,\infty)$	$[0,\infty)$
------	$(-\infty,2)\cup(2,\infty)$	$(-\infty,1)\cup(1,\infty)$

Example 1: Here are two functions, $f(x)$ and $g(x)$,



$f(x)$

$g(x)$

a. Determine $f(3)$ and $g(3)$.

$f(3)=3$ $g(3)=-2$

b. Determine $f(3)+g(3)$. Note: $f(3)+g(3) = \boxed{(f+g)(3).}$

$3+-2=1$

c. Determine $(g - f)(-5)$.

$g(-5) - f(-5)$

$-2 - 5 = -7$

d. What are the approximate zeros of $(f - g)(x)$?

$f(x) - g(x) = 0$

$x = -2.6$ & $x = 1.3$

e. What are the domain and range of $f(x)$?

D: $(-\infty, \infty)$

R: $[0, \infty)$

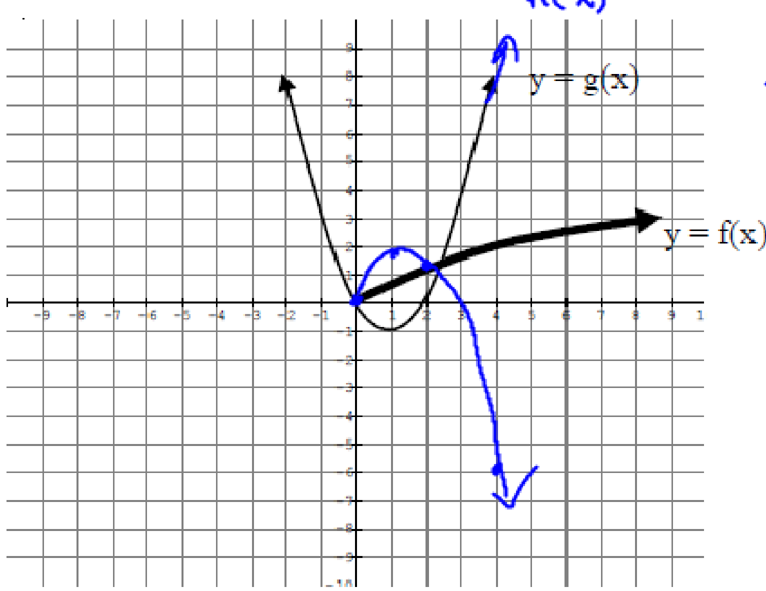
f. What are the zeros of $(f \cdot g)(x)$?

$f(x)g(x) = 0$ $\quad$ $f(x) = 0$ or $g(x) = 0$

$\boxed{x = 0 \qquad x = -4, 2}$

Example 2: Sketch the graph of $y = f(x) - g(x)$.

$h(x)$

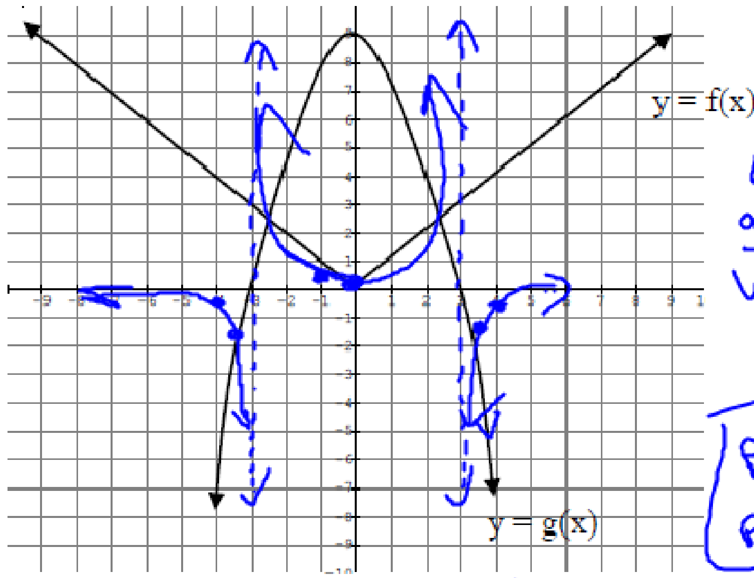


x	$f(x)$	$g(x)$	$h(x)$
-1	—	3	—
0	0	0	0
1	0.75	-1	1.75
2	1.25	0	1.25
4	2	8	-6

$\therefore (0,0), (1, 1.75), (2, 1.25)$
$(4, -6)$

Example 3: Sketch the graph of $y = \frac{f(x)}{g(x)}$. State the domain of $y = f(x), y = g(x)$, and $y = \frac{f(x)}{g(x)}$.

$y = \frac{f(x)}{g(x)}$



when $g(x) = 0$, you have a Vertical Asymptote.

$g(x) = 0$ at 3 & −3

VA @ 3 & −3

$\boxed{\text{pg 483 \#6,7} \quad \text{pg 496 \#2*,3*}}$

x	$f(x)$	$g(x)$	$\frac{f(x)}{g(x)}$
−4	4	−7	−0.6
−3.5	3.5	−2	−1.75
0	0	9	0
−1	1	8	0.1
3.5	3.5	−3	−1.2
4	4	−7	−0.6

4.2 Combining Functions Algebraically

Recall: Given $f(x) = 3 - 4x + 5x^2$, find the value of:

a. $f(2) =$

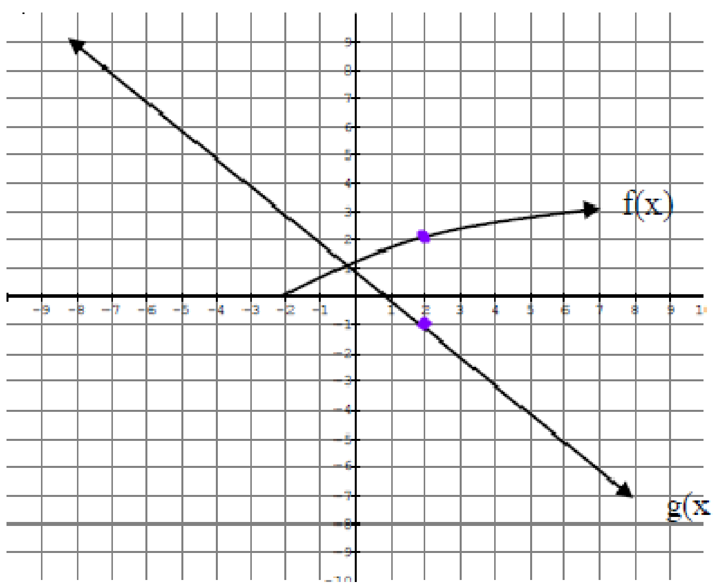
b. $f(0) =$

c. $f(-1) =$

If $f(x) = 2x - 3$ and $g(x) = -x + 2$, you can form the sum, difference, product, and quotient of f and g in the following manner:

$f(x) + g(x) = (2x - 3) + (-x + 2) = 2x - x - x + 2 = x - 1$	Sum
$f(x) - g(x) = (2x - 3) - (-x + 2) = 2x - x + x - 2 = 3x - 5$	Difference
$f(x) \cdot g(x) = (2x - 3)(-x + 2) = -2x^2 + 4x + 3x - 6 = -2x^2 + 7x - 6$	Product
$\frac{f(x)}{g(x)} = \frac{2x-3}{-x+2}$	Quotient

Example 1: Here are two functions, $f(x)$ and $g(x)$.



a. If $h(x) = f(x) + g(x)$, find the value of $h(2)$.

$= f(2) + g(2)$

$= 2 + (-1)$

$= 1$

b. If $d(x) = f(x) - g(x)$, find the value of $d(2)$.

$= f(2) - g(2)$
$= 2 - (-1)$
$= 3$

c. If $p(x) = f(x) * g(x)$, find the value of $p(2)$.

$= f(2) \times g(2)$
$= 2(-1)$
$= -2$

Example 2: If $f(x) = \sqrt{x}$ and $g(x) = x - 3$,

a. State the domain and range of $f(x)$ and $g(x)$.

	$f(x)$	$g(x)$
D:	$[0, \infty)$	$(-\infty, \infty)$
R:	$[0, \infty)$	$(-\infty, \infty)$

b. Write the explicit equation of $h(x)$, given $h(x) = f(x) + g(x)$. Find the domain and range of $h(x)$.

$h(x) = \sqrt{x} + x - 3$

$\boxed{\text{yint} = -3}$

$h(1) = \sqrt{1} + 1 - 3$
$= 1 + 1 - 3$
$= -1$

Domain: $[0, \infty)$

Range: $[-3, \infty)$

Use graphing technology to determine the range of the new function.

Note: When two functions are combined through addition, subtraction, or multiplication, the new domain is the set of x values that are common to the domains of the combined functions.

c. If $p(x) = f(x) * g(x)$, write an explicit equation. Determine the domain of the new function.

$p(x) = (\sqrt{x})(x-3)$

Domain: $[0, \infty)$

d. If $d(x) = \frac{f(x)}{g(x)}$, write an explicit equation for $d(x)$. Then determine its domain.

$d(x) = \frac{\sqrt{x}}{x-3}$ $\quad x \neq 3$

Domain: $[0,3) \cup (3, \infty)$

$\{x \mid x \geq 0, x \neq 3, x \in \mathbb{R}\}$

Note: When two functions are combined through division, the new domain is restricted to those values of x for which the function in the denominator does not equal zero and for which the two functions are defined.

Example 3: If $f(x) = x + 2$ and $g(x) = |x|$,

a. State the domain and range of $f(x)$ and $g(x)$.

	$f(x)$	$g(x)$
D:	$(-\infty, \infty)$	$(-\infty, \infty)$
R:	$(-\infty, \infty)$	$[0, \infty)$

b. Write the explicit equation of $h(x) = f(x) - g(x)$. Find the domain and range of $h(x)$.

$h(x) = x + 2 - |x|$

Domain: $(-\infty, \infty)$

Range: $(-\infty, 2]$

$h(0) = 0 + 2 - |0|$
$= 2$

$h(1) = 1 + 2 - |1|$
$= 1 + 2 - 1$
$= 2$

$h(-1) = -1 + 2 - |-1|$
$= -1 + 2 - 1$
$= 0$

c. If $p(x) = f(x) * g(x)$, write an explicit equation. Determine the domain and range of the new function.

$p(x) = (x+2)(|x|)$

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$ (Essentially the absolute value function acts like a constant multiplier.)

d. If $d(x) = \frac{f(x)}{g(x)}$, write an explicit equation for $d(x)$. Then determine its domain.

$d(x) = \frac{x+2}{|x|}$ $\quad x \neq 0$

Domain: $(-\infty, 0) \cup (0, \infty)$
$\{x \mid x \neq 0, x \in \mathbb{R}\}$

Example 4: Consider the function $h(x) = x^2 - 2x - 3$

a. Write explicit equations for three functions, $f(x), g(x)$, and $d(x)$ so that $h(x) = f(x) + g(x) + d(x)$.

$h(x) = x^2 - 2x - 3$

$f(x) = x^2$
$g(x) = -2x$
$d(x) = -3$

b. Write explicit equations for two functions, $f(x)$ and $g(x)$ so that $h(x) = f(x) - g(x)$.

$h(x) = x^2 - (2x+3)$
$\therefore f(x) = x^2 \quad g(x) = 2x+3$

c. Write explicit equations for two functions, $f(x)$ and $g(x)$, so that $h(x) = f(x) * g(x)$.

$h(x) = f(x)g(x)$
$= x^2 - 2x - 3$
$= (x-3)(x+1)$

$\therefore f(x) = x-3$
$g(x) = x+1$

Example 5: Given $q(x) = x + 1$, write explicit equations for two new functions $f(x)$ and $g(x)$ so that $q(x) = \frac{f(x)}{g(x)}$.

$x+1 = \frac{f(x)}{g(x)}$

$x+1 = \frac{(x+1)(x+1)}{x+1}$

$\therefore f(x) = x^2+2x+1$
$g(x) = x+1$

4.3 Introduction to Composite Functions

Given f and g are two functions of x, the composition of f and g is $f(g(x))$. We read both of these expressions as: "f of g at x".

Example 1

x	f(x)
-2	8
-1	3
0	0
1	-1
2	0

x	g(x)
-2	3
-1	2
0	1
1	0
2	-1

a. Find $f(g(-1))$

$g(-1) = 2$

$f(g(-1)) = f(2) = 0$

b. Find $f(f(1))$

$f(1) = -1$

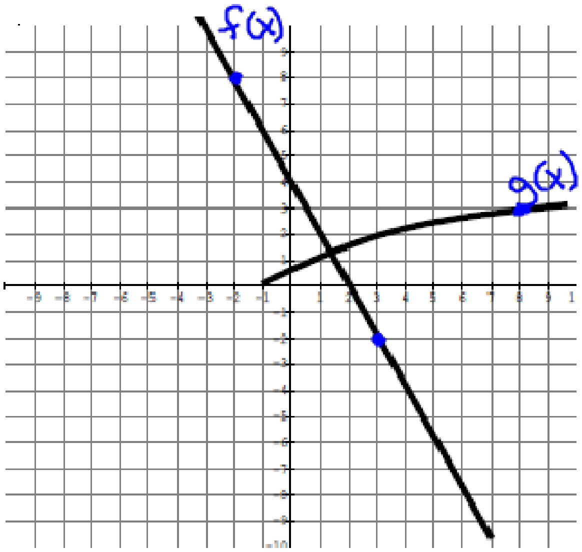
$f(f(1)) = f(-1) = 3$

c. Find $g(f(2))$

$f(2) = 0$

$g(f(2)) = g(0) = 1$

Example 2: Given the graphs of $y = f(x)$ and $y = g(x)$, determine each value below.



a. $g(f(-2))$

$f(-2) = 8$

$g(f(-2)) = g(8)$

$g(8) = 3$

b. $g(f(3))$

$f(3) = -2$

$g(f(3)) = g(-2)$

$\therefore g(f(3))$ is undefined

Example 3: Given the functions $f(x) = |x - 1|$ and $g(x) = \frac{1}{x^2+1}$, determine each value.

a. $f\big(g(2)\big)$

$g(2) = \frac{1}{2^2+1}$

$= \frac{1}{5}$

$f(g(2)) = f\left(\frac{1}{5}\right)$

$= \left|\frac{1}{5} - 1\right|$

$= \left|-\frac{4}{5}\right|$

$\boxed{= \frac{4}{5}}$

b. $g\big(f(2)\big)$

$f(2) = |2-1|$

$= |1|$

$= 1$

$\therefore g(f(2)) = g(1) = \frac{1}{1^2+1}$

$\boxed{= \frac{1}{2}}$

Example 4: If $f(x) = 2x + 3$, find the inverse. Simplify $f\big(g(x)\big)$ and $g\big(f(x)\big)$.

Inverse of $f(x)$

$x = 2y + 3$

$\frac{x-3}{2} = \frac{2y}{2}$

$\therefore y = \frac{x-3}{2}$

$g(x) = \frac{x-3}{2}$

$f(g(x))$

$f(g(x)) = \cancel{2}\left(\frac{x-3}{\cancel{2}}\right) + 3$

$= x - 3 + 3$

$= x$

$g(f(x))$

$g(f(x)) = \frac{(2x+3)-3}{2}$

$= \frac{2x}{2}$

$= x$

Note: In general, $f\big(g(x)\big)$ and $g\big(f(x)\big)$ are usually different. If $f\big(g(x)\big) = g\big(f(x)\big) = x$ for all values of x, then the two functions are inverses of each other.

Example 5: Given $f(x) = x^2 + 3x$ and $g(x) = 3x - 5$, determine an explicit equation for each composite function, ~~then state its domain.~~

a. $f(g(x))$ g(x) goes into f(x) whenever you see an x.

$(g(x))^2 + 3(g(x))$

$(3x-5)^2 + 3(3x-5)$

$(3x-5)(3x-5) + 9x - 15$

$9x^2 - 15x - 15x + 25 + 9x - 15$

$= 9x^2 - 21x + 10$

b. $g(f(x))$

$g(f(x)) = 3(f(x)) - 5$

$= 3(x^2+3x) - 5$

$= 3x^2 + 9x - 5$

c. $f(f(x))$

$f(f(x)) = (f(x))^2 + 3(f(x))$

$= (x^2+3x)^2 + 3(x^2+3x)$

$= (x^2+3x)(x^2+3x) + 3x^2 + 9x$

$= x^4 + 3x^3 + 3x^3 + 9x^2 + 3x^2 + 9x$

$= x^4 + 6x^3 + 12x^2 + 9x$

4.4 Determining Restrictions on Composite Functions

Solve:

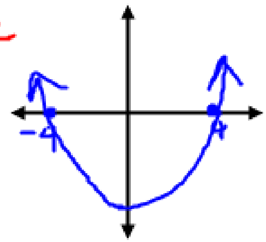
a. $x^2 + 9 \le 25$

$-25 \quad -25$

$x^2 - 16 \le 0$

$(x+4)(x-4) \le 0$

≤ 0 Want negative y-values

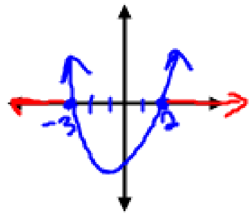


$[-4, 4]$

b. $x^2 + x - 6 > 0$

$(x+3)(x-2) > 0$

$>$ Want positive y-values



$(-\infty, -3) \cup (2, \infty)$

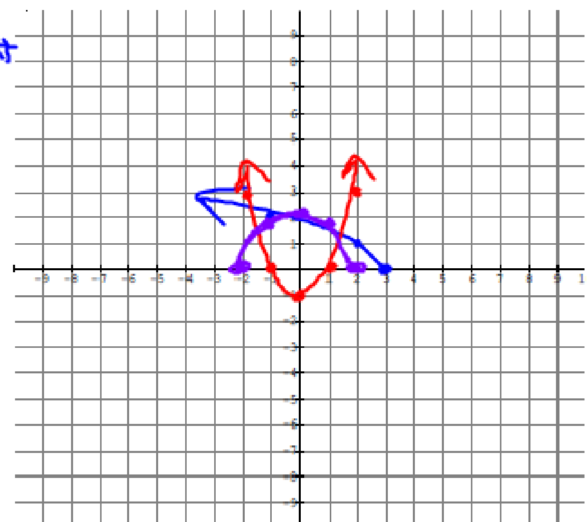
$\{x \mid x < -3, x > 2, x \in \mathbb{R}\}$

Graphing Composition Functions

Example 1: Graph $f(x) = \sqrt{3 - x}$ and $g(x) = x^2 - 1$.

$\sqrt{-(x-3)}$ — move 1 down

- reflect through y
- moves 3 to the right



a. Domain: $f(x)$

$(-\infty, 3]$

b. Range: $f(x)$

$[0, \infty)$

c. Domain: $g(x)$

$(-\infty, \infty)$

d. Range: $g(x)$

$[-1, \infty)$

e. Complete the table.

$g(x) = x^2 - 1$ $\quad$ $f(x) = \sqrt{3 - x}$ $\quad$ $f(g(x))$

x	g(x)	f(g(x))
-3	8	—
-2	3	0
-1	0	$\sqrt{3}$
0	-1	$\sqrt{3-(-1)} = \sqrt{4} = 2$
1	0	$\sqrt{3}$
2	3	0
3	4 (written over 8)	—

f. Graph $f(g(x))$

g. Domain: $f(g(x))$ and Range: $f(g(x))$

$[-2, 2]$ $\qquad$ $[0, 2]$

Example 2: Graph $f(x) = x + 1$ and $g(x) = 4 - x^2$ $= -x^2+4$

a. Domain: $f(x)$

$(-\infty, \infty)$

b. Range: $f(x)$

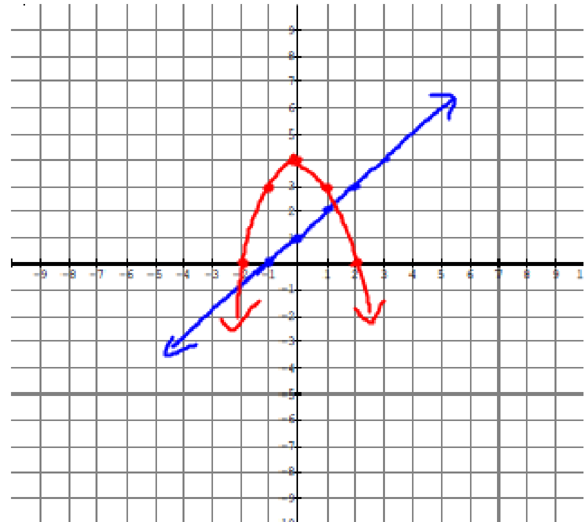
$(-\infty, \infty)$

c. Domain: $g(x)$

$(-\infty, \infty)$

d. Range: $g(x)$

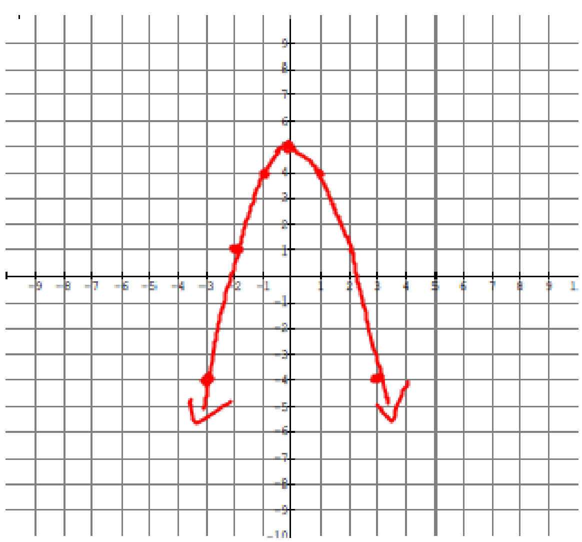
$(-\infty, 4]$



e. Complete the table. $g(x) = 4 - x^2$ $f(x) = x+1$

x	g(x)	f(g(x))
-3	-5	-4
-2	0	1
-1	3	4
0	4	5
1	3	4
2	0	1
3	-5	-4

f. Graph $f(g(x))$



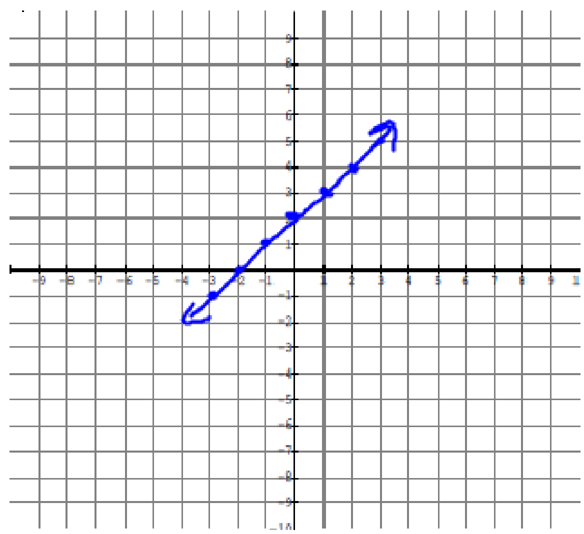
D: $(-\infty, \infty)$

R: $(-\infty, 5]$

g. Domain: $f(g(x))$ and Range: $f(g(x))$

h. Graph $f(f(x))$ $\quad$ $f(x) = x + 1$

x	f(x)	f(f(x))
-3	-2	-1
-2	-1	0
-1	0	1
0	1	2
1	2	3
2	3	4
3	4	5

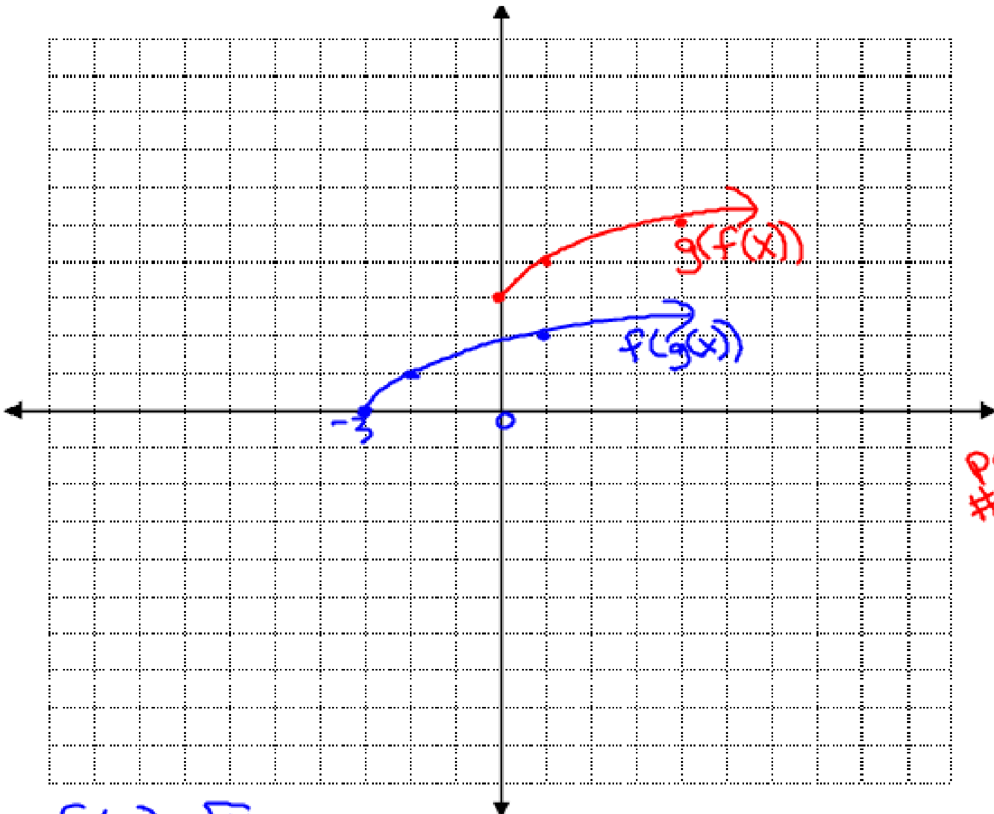


D: $(-\infty, \infty)$
R: $(-\infty, \infty)$

i. Domain: $f(f(x))$

j. Range: $f(f(x))$

Example 3: Graph $f(x) = \sqrt{x}$ and $g(x) = x + 3$.



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#4c, d, 6a,
10, 20b

$f(x) = \sqrt{x}$
$g(x) = x+3$

a. Graph $f(g(x))$ $= \sqrt{g(x)} = \sqrt{x+3}$ → move 3 to left

b. Graph $g(f(x))$ $= \sqrt{x} + 3$ → Moved up 3

For the composition function $f(g(x))$, the function f is applied to the range of g. The domain of f is restricted to $x \geq 0$, so $f(g(x))$ is defined when $g(x) \geq 0$; that is for $x \geq -3$.

For the composition function $g(f(x))$, the function g is applied to the range of f. The domain of f is restricted to $x \geq 0$, so $g(f(x))$ is defined for $x \geq 0$.

In general, $f(g(x))$ is only defined for $x = a$ when:

- a is in the domain of g, and
- $g(a)$, which is an element of the range of g, is in the domain of f

Determining a Composition Function

Example 1: Given $f(x) = \frac{1}{x+3}$ and $g(x) = x^2 - 4x$, determine an explicit equation for each composite function below, then state its domain.

a. $g(f(x))$

$$g(f(x)) = (f(x))^2 - 4(f(x))$$
$$= \left(\frac{1}{x+3}\right)^2 - 4\left(\frac{1}{x+3}\right)$$
$$= \frac{1}{(x+3)^2} - \frac{4(x+3)}{(x+3)(x+3)}$$
$$= \frac{1-4(x+3)}{(x+3)^2}$$
$$= \frac{1-4x-12}{(x+3)^2} = \frac{-4x-11}{(x+3)^2}$$

b. $f(g(x))$

$$f(g(x)) = \frac{1}{g(x)+3} = \frac{1}{x^2-4x+3}$$

Example 2: Given $f(x) = \sqrt{x}$ and $g(x) = x^2 - 4$, determine an explicit equation for each composite function below, then state its domain.

a. $g(f(x))$

$g(f(x)) = (f(x))^2 - 4$

$= (\sqrt{x})^2 - 4$

$= x - 4$

Domain: $[0, \infty)$

↳ because of $\sqrt{x}$ in original $f(x)$ function

b. $f(g(x))$

$f(g(x)) = \sqrt{g(x)} = \sqrt{x^2 - 4}$

$x^2 - 4 \geq 0$

$(x+2)(x-2) \geq 0$

Domain: $(-\infty, -2] \cup [2, \infty)$

Writing a Function as a Composition of Two Functions

Example 1: For each function, determine possible functions f and g so that $y = f(g(x))$.

a. $y = \frac{1}{x+3}$ $= f(g(x))$

$f(g(x)) = \frac{1}{g(x)} = \frac{1}{x+3}$

$f(x) = \frac{1}{x}$

$g(x) = x + 3$

b. $y = (x-2)^2$ $= f(g(x))$

$f(x) = x^2$

$g(x) = x - 2$

c. $y = \sqrt{3+x}$ $= f(g(x))$

$f(x) = \sqrt{x}$

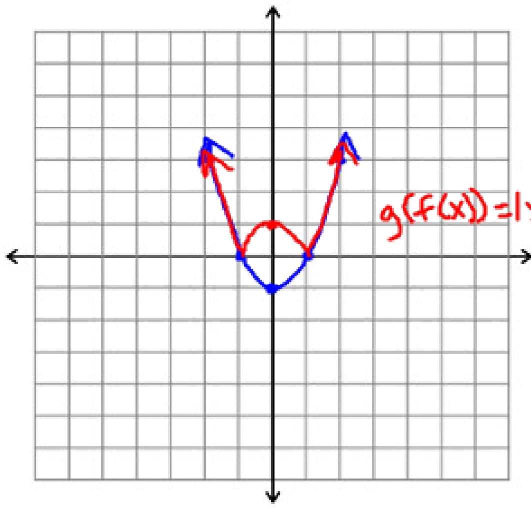
$g(x) = 3 + x$

Graphing Composition Functions

Example 1: Graph $g(f(x))$ for each function below where $g(x) = |x|$.

a. $f(x) = x^2 - 1$

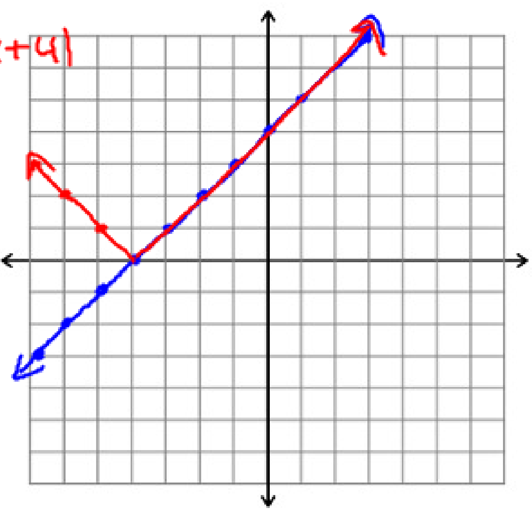
$g(f(x))$
$= |x^2 - 1|$



Graph $y = x^2 - 1$
Then reflect negative y-values over x-axis.

b. $f(x) = x + 4$

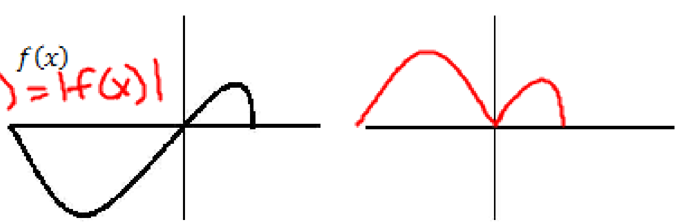
$g(f(x)) = |x+4|$



Graph $x + 4 = y$.

c. $f(x)$

$g(f(x)) = |f(x)|$



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$g(f(x)) = |f(x)|$

d. $f(x)$