Chapter 5 Exponential and Logarithmic Functions

5.1 Graphing Exponential Functions

Graph $y = 2^x$ using a table of values.

x	$y = 2^x$
-3	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	1
1	2
2	4
3	8

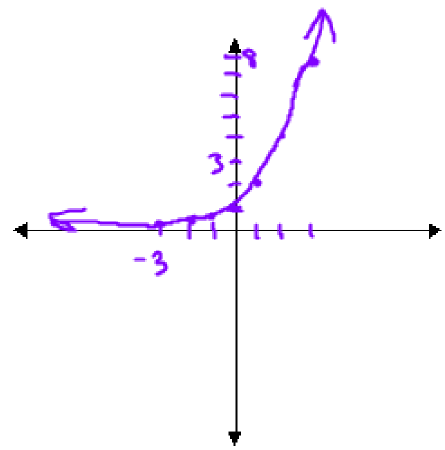
x-intercepts: none

y-intercept: $y=1$

asymptotes: $y=0$

domain: $(-\infty, \infty)$

range: $(0, \infty)$



Graph $y = \left(\frac{1}{2}\right)^x$ using a table of values.

x	$y = \left(\frac{1}{2}\right)^x$
-3	$\left(\frac{1}{2}\right)^{-3} = 2^3 = 8$
-2	$\left(\frac{1}{2}\right)^{-2} = 2^2 = 4$
-1	$\left(\frac{1}{2}\right)^{-1} = 2^1 = 2$
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$

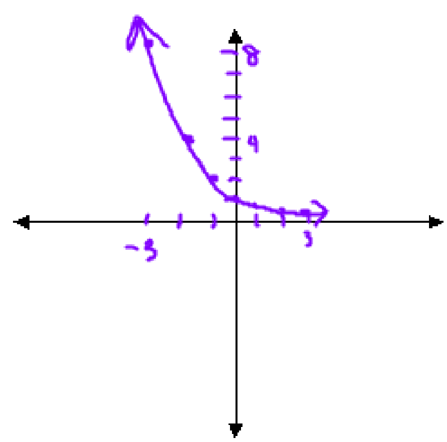
x-intercepts: none

y-intercept: $y=1$

asymptotes: $y=0$

domain: $(-\infty, \infty)$

range: $(0, \infty)$



Graph each of the following without a table of values and state all properties.

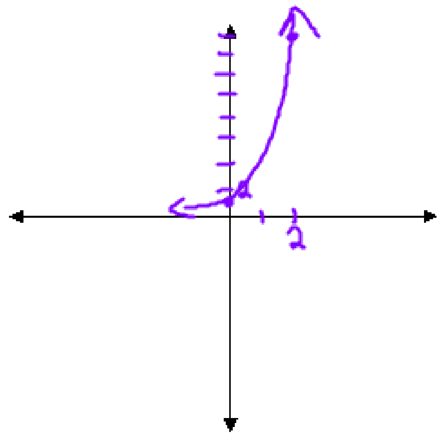
$y = 4^x$

You need 2 points to graph.

yint: $x = 0$
$y = 4^0$
$y = 1$

$x = 2$
$y = 4^2$
$y = 16$

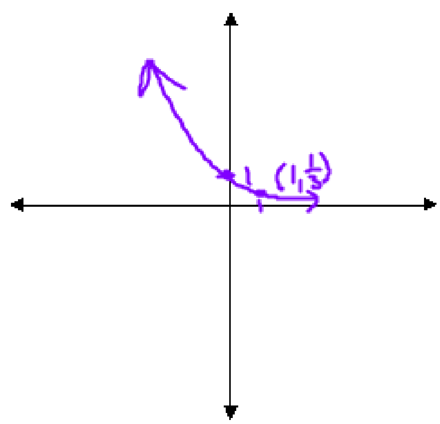
x ints = none
yint = 1
asymptote = $y = 0$
D: $(-\infty, \infty)$
R: $(0, \infty)$



$y = \left(\frac{1}{3}\right)^x$

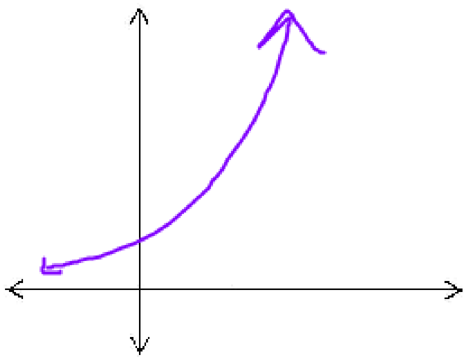
$x = 0$
$y = \left(\frac{1}{3}\right)^0$
$y = 1$
$(0, 1)$

$x = 1$
$y = \left(\frac{1}{3}\right)^1$
$y = \frac{1}{3}$
$(1, \frac{1}{3})$

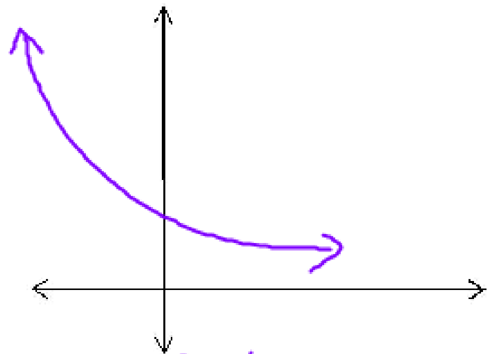


An **exponential function** is any function of x that can be written in the form $y = a^x$, where a is a positive constant.

$y = a^x$ when $a > 1$



$y = a^x$ when $0 < a < 1$



pg. 342 #1, 2a, b, 3, 5a, c, 6

Properties of $y = a^x$

- When $a > 1$, the graph is said to be **increasing**.
- When $0 < a < 1$, the graph is said to be **decreasing**.
- The graph has a y-intercept of 1.
- The graph approaches the x-axis, but never reaches it. The function has a horizontal asymptote with equation $y = 0$.
- The graph does not have an x-intercept.
- The domain of the function is $x \in R$, or $(-\infty, \infty)$
- The range of the function is $y > 0$, or $(0, \infty)$

5.2 Analyzing Exponential Functions

Consider the function $y - k = c\,a^{d(x-h)}$ $\quad y=c\left(a^{d(x-h)}\right)+k$

$a \rightarrow$ base

- c will cause a vertical stretch or compression
- d will cause a horizontal stretch or compression
- k will cause a vertical translation
- h will cause a horizontal translation
- $d < 0$ will cause a reflection over the y-axis
- $c < 0$ will cause a reflection over the x-axis

Example 1

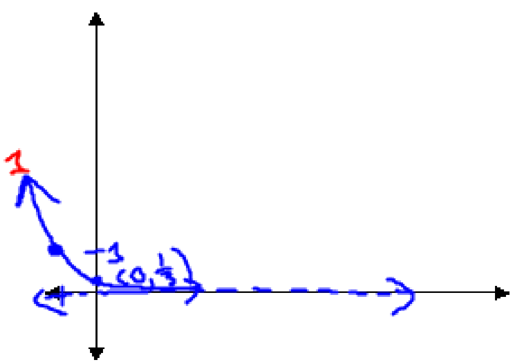
a. Graph $y = \left(\frac{1}{3}\right)^{x+1}$

$y=\left(\frac{1}{3}\right)^{x} \rightarrow y=\left(\frac{1}{3}\right)^{x+1}$

Moved to left 1

$(x, y) \rightarrow (x-1, y)$

(x, y)	$(x-1, y)$
$(0, 1)$	$(-1, 1)$
$(1, \frac{1}{3})$	$(0, \frac{1}{3})$



b. Determine:

i. whether the function is increasing or decreasing

ii. the intercepts

y int $= \frac{1}{3}$

iii. the equations of any asymptotes

$y=0$

iv. the domain of the function

$(-\infty, \infty)$

v. the range of the function

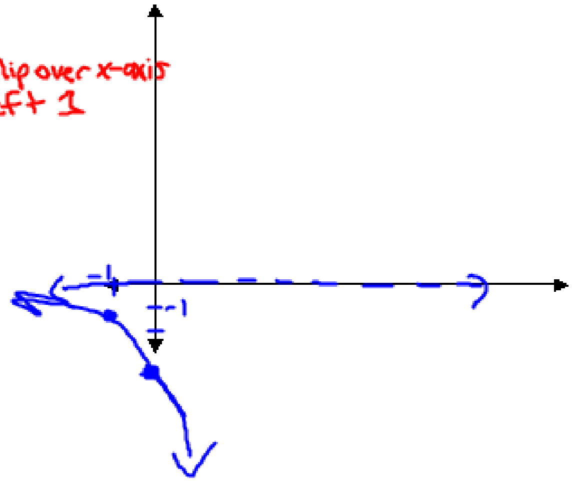
$(0, \infty)$

Example 2

a. Graph $y = -(3)^{x+1}$

$y = 3^x$ $\leftrightarrow$ $y = -(3)^{x+1}$ — flip over x-axis — left 1

(x, y)	$(x-1, -y)$
$(0, 1)$	$(-1, -1)$
$(1, 3)$	$(0, -3)$



b. Determine:

i. whether the function is increasing or decreasing

ii. the intercepts y int = -3

iii. the equations of any asymptotes $y = 0$

iv. the domain of the function $(-\infty, \infty)$

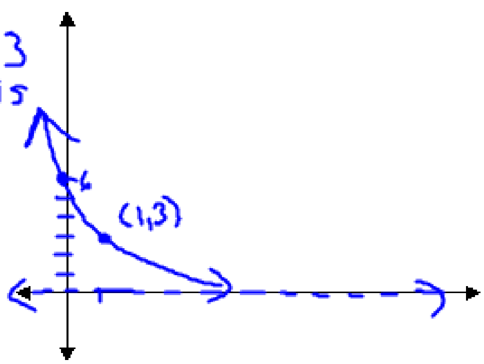
v. the range of the function $(-\infty, 0)$

Example 3

a. Graph $y = 3(2)^{-(x-1)}$

$y = 2^x$ $\leftrightarrow$ — vertical stretch by 3 — flip through y-axis — moves right 1

(x, y)	$(-x+1, 3y)$
$(0, 1)$	$(1, 3)$
$(1, 2)$	$(0, 6)$



b. Determine:

i. whether the function is increasing or decreasing

ii. the intercepts $y = 6$

iii. the equations of any asymptotes $y = 0$

iv. the domain of the function $(-\infty, \infty)$

v. the range of the function $(0, \infty)$

$y = 2^{0-2} - 1 \quad = \frac{1}{4} - 1$
$y = 2^{-2} - 1 \quad = -0.75$
$y = \frac{1}{2^2} - 1$

Example 4:

a. Graph $y + 1 = (2)^{x-2}$

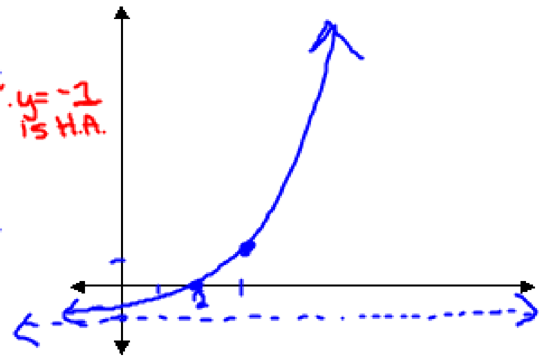
$y = 2^x$ $\quad y = 2^{(x-2)} - 1$

- down 1 $\quad \therefore y = -1$ is H.A.
- right 2

$(x, y) \rightarrow (x+2, y-1)$

$(0, 1) \rightarrow (2, 0)$

$(1, 2) \rightarrow (3, 1)$



b. Determine:

i. whether the function is increasing or decreasing

ii. the intercepts x int = 2, y int = −0.75

iii. the equations of any asymptotes $y = -1$

iv. the domain of the function $(-\infty, \infty)$

v. the range of the function $(-1, \infty)$

$y = 3(2^{-0+2})$
$y = 3(2^2)$
$y = 3(4)$
$y = 12$

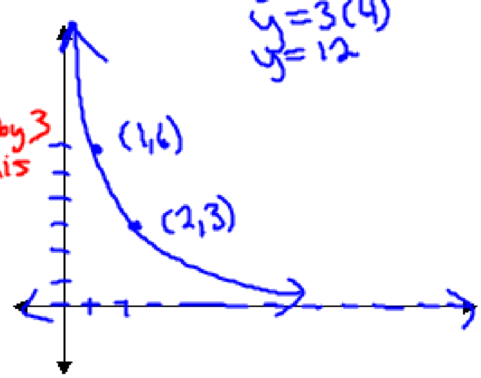
Example 5

a. Graph $y = 3(2^{-x+2})$

$y = 2^x$ $\quad y = 3(2^{-(x-2)})$

- vertical stretch by 3
- reflect in y-axis
- 2 to right

(x, y)	$(-x+2, 3y)$
$(0, 1)$	$(2, 3)$
$(1, 2)$	$(1, 6)$



b. Determine:

i. whether the function is increasing or decreasing

ii. the intercepts x = 0 to find y. y = 12

iii. the equations of any asymptotes $y = 0$

iv. the domain of the function $(-\infty, \infty)$

v. the range of the function $(0, \infty)$

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#1, 3a, b, 4, 5, 7b, c, 9

5.3 Solving Exponential Equations

An **exponential equation** contains a power with a variable in the exponent.

One strategy to solve an exponential equation is to use a graphing calculator and type in each side of the equation and find where the curves intersect.

The other strategy to solve an exponential equation is to make the bases the same and then equate the exponents. $x^a x^b = x^{a+b}$ $\frac{x^a}{x^b} = x^{a-b}$ $(x^a)^b = x^{ab}$

Example 1: Solve each equation. $\sqrt{x} = x^{\frac{1}{2}}$ $\sqrt[3]{x} = x^{\frac{1}{3}}$ $\sqrt[4]{x} = x^{\frac{1}{4}}$

a. $16^{2x-3} = 32^{x+3}$

Make bases the same.

$2^4 = 16$ $2^5 = 32$

$2^{4(2x-3)} = 2^{5(x+3)}$ Equate your exponents.

$4(2x-3) = 5(x+3)$

$8x - 12 = 5x + 15$

$-5x \ +12 \quad -5x \ +12$

$3x = 27$ $\boxed{\therefore x = 9}$

b. $27^x = 9^{2x-1}$

$3^3 = 27$ $3^2 = 9$

$3^{3(x)} = 3^{2(2x-1)}$ Equate exponents.

$3x = 2(2x-1)$

$3x = 4x - 2$

$-4x \quad -4x$

$-x = -2$

$\boxed{x = 2}$

c. $4^x = \frac{1}{256}$ $256 = 4^4$

$4^x = \frac{1}{4^4}$

$4^x = 4^{-4}$

$x = -4$

$2^{-1} = \frac{1}{2^1}$

$2^{-4} = \frac{1}{2^4}$

Example 2: Solve each equation.

a. $2^x = 8\sqrt[3]{2}$

$\sqrt[3]{2} = 2^{\frac{1}{3}}$

$8 = 2^3$

$2^x = 2^3(2^{\frac{1}{3}})$

Add $3 + \frac{1}{3}$

$\frac{9}{3} + \frac{1}{3} = \frac{10}{3}$

$2^x = 2^{\frac{10}{3}}$

$\boxed{\therefore x = \frac{10}{3}}$

b. $\left(\sqrt{125}\right)^{2x+1} = \sqrt[3]{625}$ Get rid of any roots.

$\sqrt{125} = 125^{\frac{1}{2}}$ $\sqrt[3]{625} = 625^{\frac{1}{3}}$

$125^{\frac{1}{2}(2x+1)} = 625^{\frac{1}{3}}$

$125^{x+\frac{1}{2}} = 625^{\frac{1}{3}}$ $125 = 5^3$ $625 = 5^4$

$5^{3(x+\frac{1}{2})} = 5^{4(\frac{1}{3})}$

$3(x+\frac{1}{2}) = 4(\frac{1}{3})$

$3x + \frac{3}{2} = \frac{4}{3}$

$3x = \frac{4}{3} - \frac{3}{2}$

$\frac{4}{3} - \frac{3}{2}$

$\frac{8}{6} - \frac{9}{6} = -\frac{1}{6}$

$3x = -\frac{1}{6}$

$x = -\frac{1}{18}$

$x = \frac{-\frac{1}{6}}{3} = -\frac{1}{6} \div \frac{3}{1} = -\frac{1}{6} \times \frac{1}{3} = -\frac{1}{18}$

c. $\frac{81^{3n+2}}{243^{-n}} = 3^4$

$81 = 3^4$

$243 = 3^5$

$\frac{3^{4(3n+2)}}{3^{5(-n)}} = 3^4$

$\frac{3^{12n+8}}{3^{-5n}} = 3^4$

$3^{12n+8-(-5n)} = 3^4$

$12n + 8 + 5n = 4$

$17n + 8 - 8 = 4 - 8$

$17n = -4$

$n = \frac{-4}{17}$

Take a guess at what these statements are saying:

$power_2(8) = 3$ $2^3 = 8$

$power_2(32) = 5$ $2^5 = 32$

$power_3(9) = 2$ $3^2 = 9$

Then, see if you can fill in the blanks:

$power_2(16) = 4$ $2^4 = 16$

$power_6(36) = 2$

$power_5(125) = 3$ $5^3 = 125$

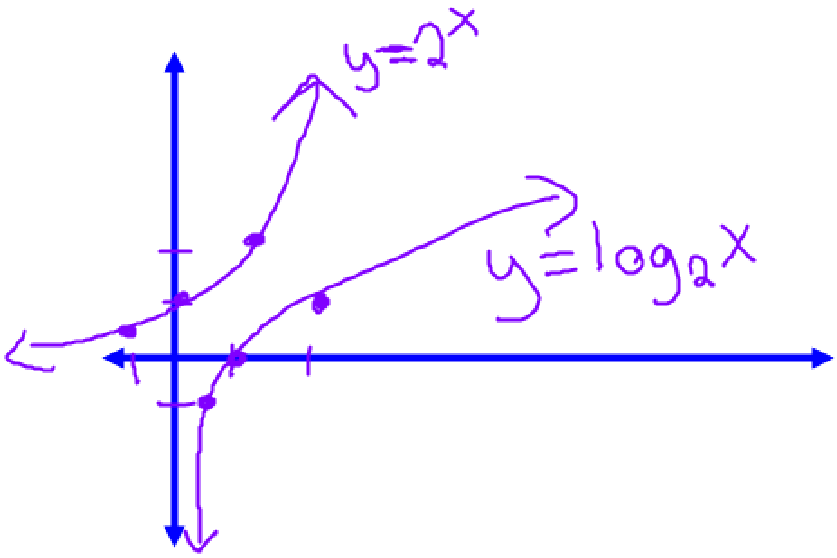
$y = \log_2 x$

$x = 2^y$

Inverse

5.4 Logarithms and the Logarithmic Function

Graph $y = 2^x$ and its inverse on the same plane.



$y=2^x$

x	y
-1	$\frac{1}{2}$
0	1
1	2

Inverse

x	y
$\frac{1}{2}$	-1
1	0
2	1

The term **logarithm** is used to describe the inverse of a power. The inverse of 10^x is the logarithm to the base 10 of x or $\log_{10} x$.

The logarithm of a number is an exponent.
$\log_b c$ is the power to which b is raised to get c.

The base of the logarithm is the same as the base of the power.
$\boxed{\text{When } \log_b c = a, \text{ then } c = b^a,}$ where $b > 0, b \neq 1, c > 0$.

$\log_b c = a$ $\quad$ $b^a = c$

Example 1:

$\boxed{\text{When } \log_b c = a, \text{ then } c = b^a}$

a. Write each exponential expression as a logarithmic expression.

i. $3^3 = 27$

$\log_3 27 = 3$

ii. $5^{-2} = \frac{1}{25}$ $\log_5 \frac{1}{25} = -2$

iii. $4^0 = 1$ $\log_4 1 = 0$

b. Write each logarithmic expression as an exponential expression.

i. $\log_7 49 = 2$

$7^2 = 49$

ii. $\log_4 \left(\frac{1}{64}\right) = -3$

$4^{-3} = \frac{1}{64}$

iii. $\log_{10} \left(\frac{1}{10000}\right) = -4$

$10^{-4} = \frac{1}{10000}$

$\log_{10} x$ is called the **common logarithm** of x because the base is 10. The 10 is often omitted. Therefore, $\log_{10} x$ and $\log x$ are equivalent.

Our calculators operate on base 10. $\log_{10} x = \log x$

$\log_{10} 100 = 2$

Example 2: Evaluate with a calculator.

a. $\log_{10} 5$

0.699

$\log_2 6$ ← Can't enter in calculator!

b. $\log_{10} 24$

1.380

c. $\log 46$

1.663

Consider $\log_b b^n = n$.

Example 3: Evaluate each logarithm.

a. $\log_5 3125 = 5$

$5^5 = 3125$

b. $\log_6(216) = 3$

$6^3 = 216$

c. $\log_8(2\sqrt[3]{2})$

$2^1 2^{\frac{1}{3}} = 2^{\frac{4}{3}}$

$8^x = 2^{\frac{4}{3}}$

$2^{3x} = 2^{\frac{4}{3}}$

$3x = \frac{4}{3}$

$\boxed{x = \frac{4}{9}}$

Example 4: To the nearest tenth, estimate the value of $\log_5 100$.

$5^x = 100$

$5^3 = 125$

$5^{2.9} = 106.4$

$5^{2.8} = 90.6$

$\boxed{\therefore \log_5 100 \approx 2.9}$

Example 5:

a. Graph $y = \log_4 x$.

$\log_4 x = y$

$4^y = x$

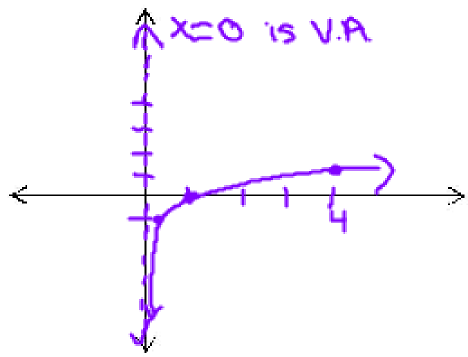
Graph the inverse function $y = 4^x$.

$y = 4^x$

x	y
-1	$\frac{1}{4}$
0	1
1	4

$y = \log_4 x$

x	y
$\frac{1}{4}$	-1
1	0
4	1



b. Identify the intercepts, the equations of any asymptotes, and the domain and range of the function.

x int → @ 1

y int → none

Asymptote → $x=0$

D: $(0, \infty)$

R: $(-\infty, \infty)$

5.5 The Laws of Logarithms

When $x > 0$ and $y > 0$, the following are true:

Product Law	$\log_b xy = \log_b x + \log_b y$	$b > 0, b \neq 1$
Quotient Law	$\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$	$b > 0, b \neq 1$
Power Law	$\log_b x^k = k \log_b x$	$b > 0, b \neq 1, k \varepsilon R$

Example 1: Simplify each expression. Use a calculator to verify the answer.

a. $\log 7 + \log 8$ $= \log((7)(8)) = \log 56$

b. $5 \log 2$ $= \log 2^5 = \log 32$

c. $\log 80 - \log 16$ $= \log\left(\frac{80}{16}\right) = \log 5$

d. $\frac{1}{2} \log 100$ $= \log 100^{\frac{1}{2}} = \log \sqrt{100} = \log 10 = \log_{10} 10 = \boxed{1}$

e. $-2 \log 6$ $= \log 6^{-2} = \log\left(\frac{1}{6^2}\right) = \log\left(\frac{1}{36}\right)$

Example 2: Write each expression as a single logarithm.

a. $\log x + 3 \log y$

$= \log x + \log y^3$

$= \log(xy^3)$

b. $\log x + 2 \log y - 4 \log z$

$\log x + \log y^2 - \log z^4$ $= \log(xy^2) - \log z^4$

$= \log\left(\frac{xy^2}{z^4}\right)$

c. $\log_2 6 - 3$

You can't write 3 as an exponent with a base of 2 $\therefore$ can't simplify

$2^x = 3$
What is x?
Don't know!

d. $\log(x+1) - 3\log(y-2) - 2\log z$

$\log(x+1) - \log(y-2)^3 - \log z^2$

$= \log\left(\frac{(x+1)}{(y-2)^3}\right) - \log z^2$

$= \log\left(\frac{(x+1)}{(y-2)^3 z^2}\right)$

e. $\frac{1}{2}\log x - 2\log y - \frac{1}{7}\log z + 5\log a$

$\log x^{\frac{1}{2}} - \log y^2 - \log z^{\frac{1}{7}} + \log a^5$

$= \log\sqrt{x} - \log y^2 - \log\sqrt[7]{z} + \log a^5$

$= \log\left(\frac{\sqrt{x}}{y^2}\right) - \log\sqrt[7]{z} + \log a^5 \rightarrow = \log\left(\frac{\sqrt{x}\, a^5}{y^2\sqrt[7]{z}}\right)$

Example 3: Write each expression in terms of $\log a$, $\log b$, and/or $\log c$.

a. $\log\left(\frac{a}{b^2}\right)$

$= \log a - \log b^2$

$= \log a - 2\log b$

$i^2 = -1$

b. $\log\left(\frac{a^2 b^{\frac{1}{3}}}{c}\right)$ $= \log a^2 + \log b^{\frac{1}{3}} - \log c$

$= 2\log a + \frac{1}{3}\log b - \log c$

c. $\log\left(\frac{(x+1)^2}{(m+1)^3(n-3)^4}\right)$

$= \log(x+1)^2 - \log(m+1)^3 - \log(n-3)^4$

$= 2\log(x+1) - 3\log(m+1) - 4\log(n-3)$

→ Find a numerical answer!

Example 4: Evaluate each expression.

a. $3\log_9 6 - \log_9 72$

$\begin{array}{r} 36 \\ \times\ 6 \\ \hline 216 \end{array}$

$\begin{array}{r} 72 \\ \times\ 3 \\ \hline 216 \end{array}$

$= \log_9 6^3 - \log_9 72$

$= \log_9 216 - \log_9 72$

$= \log_9\left(\frac{216}{72}\right)$

$= \log_9 3$

$= \frac{1}{2}$

$9^x = 3$

$9^{\frac{1}{2}} = 3$

$\sqrt{9} = 3$

b. $2\log_4 6 - 3\log_4 3 + \log_4 12$

$= \log_4 6^2 - \log_4 3^3 + \log_4 12$

$= \log_4 36 - \log_4 27 + \log_4 12$

$= \log_4\left(\frac{36}{27}\right) + \log_4 12$

$= \log_4\left(\frac{36(12)}{27}\right)$

$= \log_4(16)$

$= 2$

$36 = 2\times18$

$= 2\times2\times9$

$= 2\times2\times3\times3$

$12 = 2\times6$

$= 2\times2\times3$

$\frac{(2\times2\times\cancel{3}\times\cancel{3})\times(2\times2\times\cancel{3})}{\cancel{3}\times\cancel{3}\times\cancel{3}}$

5.6 Analyzing Logarithmic Functions

When the base of a logarithm is 10 we simply type the number into a calculator to evaluate it.

For example: $\log_{10} 4 =$ 0.602 and $\log_{10} 34 =$ 1.531

When the base is not 10 we will use the **change of base** formula.

$$\boxed{\log_b x = \frac{\log x}{\log b}}$$

Example 1: Approximate the value of each logarithm, to the nearest thousandth. Write the related exponential expression.

a. $\log_5 50$ $= \dfrac{\log 50}{\log 5} = 2.43$

b. $\log_8 6$ $= \dfrac{\log 6}{\log 8} = 0.862$

Consider the function $y - k = c\log_a d(x - h)$

$y = c\log_a(d(x-h)) + K$

- c will cause a vertical stretch or compression
- d will cause a horizontal stretch or compression
- k will cause a vertical translation
- h will cause a horizontal translation
- $d < 0$ will cause a reflection over the y-axis
- $c < 0$ will cause a reflection over the x-axis

Example 2:

a. Create a table of values for $= \log_2 x$.
b. How is the graph of $y = \log_2 2x - 1$ related to the graph of $y = \log_2 x$? Sketch these two graphs on the same grid.
c. Identify the intercepts and the equation of the asymptote of the graph of $= \log_2 2x - 1$, and the domain and range of the function.

$y = 2^x$

x	y
-1	$\frac{1}{2}$
0	1
1	2

$y = \log_2 x$

x	y
$\frac{1}{2}$	-1
1	0
2	1

$y = \log_2 2x - 1$

$(\frac{1}{2}x, y-1)$

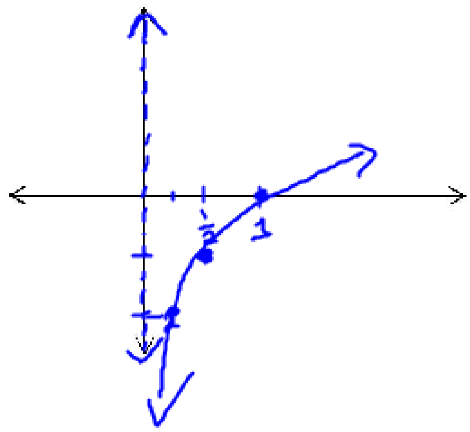
x	y
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0

x int = 1

D = $(0, \infty)$

R = $(-\infty, \infty)$

VA @ x=0



Example 3: Graph $y = -\log_2(x-1)$.

$y = 2^x$

x	y
-1	$\frac{1}{2}$
0	1
1	2

$y = \log_2 x$

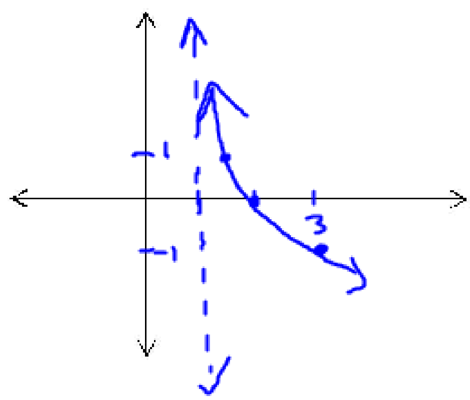
x	y
$\frac{1}{2}$	-1
1	0
2	1

$y = -\log_2(x-1)$

$(x+1, -y)$

x	y
$\frac{3}{2}$	1
2	0
3	-1

V.A. @ x=1



Example 4: Graph $y + 2 = \log_2(-x)$.

$y = 2^x$

x	y
-1	$\frac{1}{2}$
0	1
1	2

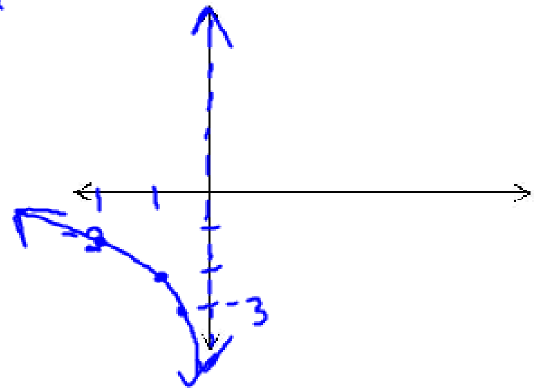
$y = \log_2 x$

x	y
$\frac{1}{2}$	-1
1	0
2	1

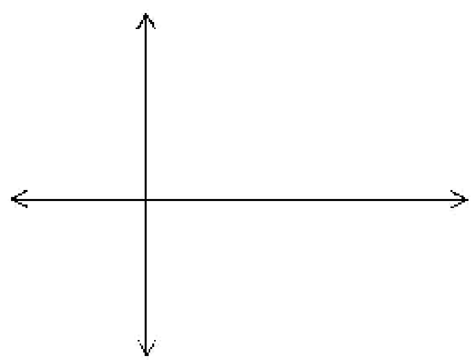
$y = \log_2(-x) - 2$

$(-x, y-2)$

x	y
$-\frac{1}{2}$	-3
-1	-2
-2	-1



Example 5: Graph $y - 3 = \log_2(x + 2)$.



5.7 Solving Logarithmic and Exponential Equations

A **logarithmic equation** is an equation that contains the logarithm of a variable.

Type 1: Logs on one side of the equation.

$\log A + \log B = \log(AB)$

$\log A - \log B = \log\left(\frac{A}{B}\right)$

$\log A^B = B \log A$

Steps: Condense the logs.

Change to exponential form.

Solve and check your solution.

Example 1: Solve: $\log_3 9x + \log_3 x = 4$. Verify the solution.

$\log_3(9x(x)) = 4$

$\log_3(9x^2) = 4$

$3^4 = 9x^2$

$\frac{81}{9} = \frac{9x^2}{9}$

$9 = x^2$

$\therefore x = \sqrt{9}$

$x = 3$ or -3

Check

$x = 3$ ✓

$x = -3$ is an "extraneous root" b/c a log value can never be negative

Example 2: Solve: $3 = \log_2(x + 2) + \log_2 x$. Verify the solution.

$3 = \log_2((x+2)(x))$

$3 = \log_2(x^2 + 2x)$

$2^3 = x^2 + 2x$

$8 = x^2 + 2x$

$0 = x^2 + 2x - 8$

$0 = (x+4)(x-2)$

$\therefore x = -4, 2$

Check

$\cancel{x = -4}$

$-4 + 2 = -2$ ✗

$\boxed{x = 2}$

$2 + 2 = 4$ ✓

Type 2: Logs on both sides of the equation.

Steps: Condense the logs.

Equate the numbers.

Solve and check your solution.

Example 3: Solve: $\log 6x = \log(x+6) + \log(x-1)$. Verify the solution.

$\log 6x = \log((x+6)(x-1))$

$\log 6x = \log(x^2 - x + 6x - 6)$

$6x = x^2 + 5x - 6$

$0 = x^2 - x - 6$

$0 = (x-3)(x+2)$

$\therefore x = 3, -2$

Check

$x = 3$ ✓

$\cancel{x=-2}$ X

$\therefore$ Only solution is $x = 3$

Example 4: Solve: $\log_5(x-18) - \log_5 x = \log_5 7$. Verify the solution.

$\log_5\left(\frac{x-18}{x}\right) = \log_5 7$

$\cancel{x}\left(\frac{x-18}{\cancel{x}}\right) = 7(x)$

$x - 18 = 7x$

$-x \qquad\quad -x$

$\frac{-18}{6} = \frac{6x}{6}$

$-3 = x$

$\therefore$ No solution b/c $x=-3$ is an extraneous root.

Type 3: No logs in the equation and the bases cannot be made equal.

Steps: Take the common log of both sides.

Expand using log laws.

Solve and check your solution.

Example 5: Solve: $12 = 4^x$

Cannot convert values to have the same base

$\therefore$ use logs

$\log 12 = \log(4^x)$

$\frac{\log 12}{\log 4} = \frac{x \cancel{\log 4}}{\cancel{\log 4}}$

$x = \frac{\log 12}{\log 4}$

$\boxed{x = 1.792}$

Example 6: Solve: $3^{x+1} = 6^x$

$\log(3^{x+1}) = \log(6^x)$

$(x+1)\log 3 = x \log 6$

$x \log 3 + \log 3 = x \log 6$

$x\log 3 - x \log 6 = -\log 3$

$\frac{x \cancel{(\log 3 - \log 6)}}{\cancel{(\log 3 - \log 6)}} = \frac{-\log 3}{(\log 3 - \log 6)}$

$x = \frac{-\log 3}{\log 3 - \log 6}$

$x = 1.585$

Example 7: Solve: $36 = 3(2^{x+1})$

$\log 36 = \log(3(2^{x+1}))$

$\log 36 = \log 3 + \log(2^{x+1})$

$\log 36 - \log 3 = (x+1)\log 2$

$\log 36 - \log 3 = x\log 2 + \log 2$

$\frac{\log 36 - \log 3 - \log 2}{\log 2} = \frac{x\cancel{\log 2}}{\cancel{\log 2}}$

$\boxed{2.585 = x}$

Applications of LOGS:

When money is invested in a savings account, it earns interest. If the interest is reinvested in the account, it also earns interest. This is called **compound interest**.

- compounded annually means compounded one time a year
- compounded semi-annually means compounded 2 times a year
- compounded quarterly means compounded 4 times a year
- compounded monthly means compounded 12 times per year

Compound Interest

$A = P\left(1 + \frac{r}{n}\right)^{nt}$ or $\cancel{A = A_0\left(1 + \frac{i}{n}\right)^{nt}}$

4% = 0.04

where

- A = final amount
- P and $\cancel{A_0}$ = initial amount
- r and $\cancel{i}$ = rate → as a decimal
- n = number of compounds in a year
- t = total time in years
- $1 + \frac{r}{n}$ and $\cancel{1 + \frac{i}{n}}$ is called the **growth factor**

Example 3: A principal of \$1500 is invested at 4% annual interest, compounded quarterly. After 7 years what is the value of the investment?

P = \$1500
r = 4% = 0.04
n = 4
t = 7

$A = P\left(1 + \frac{r}{n}\right)^{nt}$

$A = 1500\left(1 + \frac{0.04}{4}\right)^{4(7)}$

$A = 1500(1.01)^{28}$

$A = \$1981.94$

Example 4: A principal of \$2500 is invested at 2.25% annual interest, compounded monthly. To the nearest quarter of a year, when will the amount be \$2800?

$A = P(1+\frac{r}{n})^{nt}$

$P = 2500$
$r = 2.25\% = 0.0225$
$n = 12$
$t = ?$
$A = 2800$

$2800 = 2500(1 + \frac{0.0225}{12})^{12t}$

$\frac{2800}{2500} = \frac{\cancel{2500}(1.001875)^{12t}}{\cancel{2500}}$

$1.12 = (1.001875)^{12t}$

$\log 1.12 = \log(1.001875)^{12t}$

$\frac{\log 1.12}{12\log(1.001875)} = \frac{12t \log(1.001875)}{12\log(1.001875)}$

$\frac{\log 1.12}{12\log(1.001875)} = t$

$5.09 = t$

$\therefore t = 5$ years.

The compound interest formula is an example of **exponential growth**.

A function that models **exponential growth** is one of the form $y = ak^{bx}$ where $k^b > 1$.

k is called the **growth factor**.

A function that models **exponential decay** is one of the form $y = ak^{bx}$ where $0 < k^b < 1$.

k is called the **decay factor**.

Example 5: If the cabin pressure in an airplane is less than 70 kPa, passengers can suffer altitude sickness. To the nearest kilometre, at what altitude is the atmospheric pressure 70 kPa?

$P = 101.3(0.88)^h$, where h is the altitude and P is the atmospheric pressure.

$h = ?$
$P = 70$

$\frac{70}{101.3} = \frac{\cancel{101.3}(0.88)^h}{\cancel{101.3}}$

$\frac{70}{101.3} = 0.88^h$

$\log\left(\frac{70}{101.3}\right) = \log(0.88^h)$

$\frac{\log\left(\frac{70}{101.3}\right)}{\log(0.88)} = \frac{h \cancel{\log(0.88)}}{\cancel{\log(.88)}}$

$2.89 = h$

$\therefore h = 3$ km

5.8 Solving Problems with Exponents and Logarithms

When a series of equal investments is made at equal time intervals, and the compounding period for the interest is equal to the time interval for the investments, the amount in dollars, or future value FV, of these investments can be determined using this formula.

$$FV = \frac{R[(1+i)^n - 1]}{i}$$

where R is the amount of the regular investment

i is the interest rate per compounding period

n is the number of investments

Example 1: Determine how many monthly investments of \$200 would have to be made into an account that pays 6% annual interest, compounded monthly, for the future value to be \$100 000.

Many people borrow money to finance a purchase. A loan is usually repaid by making regular equal payments for a fixed period of time. The amount borrowed is called the present value, PV, of a loan. When a series of equal payments is made at equal time intervals, and the compounding period for the interest is equal to the time interval between the payments, the following formula can be used to determine the amount remaining to be paid on a loan.

$$PV = \frac{R[1 - (1 + i)^{-n}]}{i}$$

Example 2: A person borrows \$15 000 to buy a car. The person can afford to pay \$300 a month. The loan will be repaid with equal monthly payments at 6% annual interest, compounded monthly. How many monthly payments will the person make?

When physical quantities have a large range of values, they are measured using a logarithmic scale. Some examples include the Richter scale, the decibel scale, and the pH scale.

To calculate the magnitude of an earthquake, use this formula.

$$\boxed{M = \log\left(\frac{I}{S}\right)}$$

where M is the magnitude

I is the intensity of the vibrations in microns measured on a seismograph that is 100 km away from the epicentre of the earthquake

S is the intensity of a standard earthquake which has a seismograph reading of 1 micron and can barely be detected

Each increase in 1 on the logarithmic scale represents a 10-fold increase in the intensity of the quake.

Example 3: The most intense earthquake ever recorded was in Chile in May 1960, with a magnitude of 9.5.

a. Calculate the intensity of the earthquake in Chile in terms of a standard earthquake.

$M = 9.5 \quad I = ?$

$9.5 = \log\left(\frac{I}{1}\right)$

$9.5 = \log I - \log 1$

$9.5 + \overset{0}{\log 1} = \log I$

$9.5 = \log I$

$10^{9.5} = I$

$\boxed{I = 3162277660}$

b. How many times as intense as the Haiti earthquake (magnitude 7.0) was the Chile earthquake? Give the answer to the nearest whole number.

.

5.9 Natural Logarithms

Any positive number not equal to 1 can be a base for a system of logs.

Base 2 $\log_2 8 = 3$ $\log_{10} 10 = 1$

Base 4 $\log_4 16 = 2$

Base 7 $\log_7 49 = 2$

The common logarithm of a number is the logarithm of a number to the base 10.
The **natural logarithm** of a number is the logarithm of that number to the base e. The constant e is a nonrepeating, nonterminating decimal number (irrational).

e is derived from the calculus.

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$n = 10, e = 2.5937$
$n = 100, e = 2.7048$
$n = 1000. e = 2.7169$
$n = 10000, e = 2.7181$

$$\boxed{e = 2.718}$$

$\log_e = \ln$

Example 3: Evaluate.

a. $\log_e 5.36 = \ln 5.36 =$ 1.678...

b. $\ln 0.25 =$ −1.386...

c. $\ln e =$ 1

Example 4: Solve for x.

a. $\ln x = 4.1$ $x = 60.34$

$e^{4.1} = x$

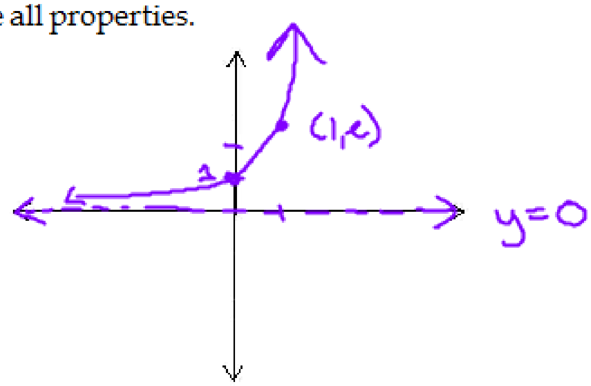
b. $\ln x = 0.89$

$e^{0.89} = x$ $x = 2.44$

Graphs of $y = e^x$. State all properties.

$y = e^x$

x	y
0	1
1	$e = 2.718$



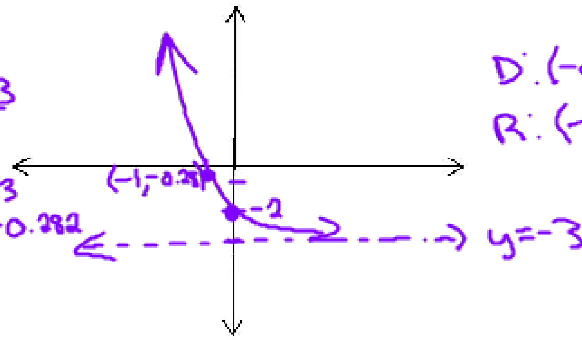
$y = e^{-x} - 3$

$y = e^x$

x	y
0	1
1	e

$-x$	$y-3$
0	-2
-1	$e-3$

≈ -0.282



D: $(-\infty, \infty)$

R: $(-3, \infty)$

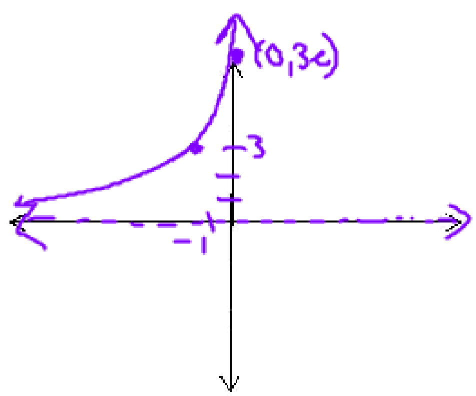
$y = 3e^{x+1}$

$y = e^x$

x	y
0	1
1	e

$x-1$	$3y$
-1	3
0	$3e$

≈ 8.155



Graphs of $y = \ln x$. State all properties.

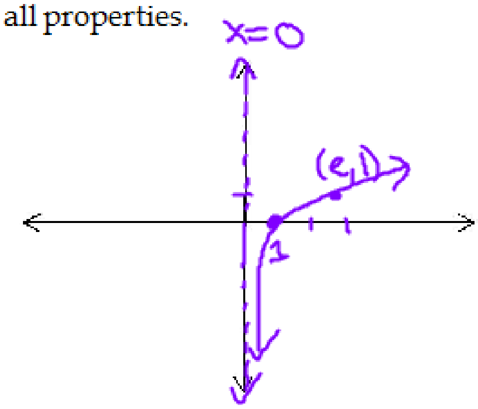
$y = \ln x$

$y = e^x$

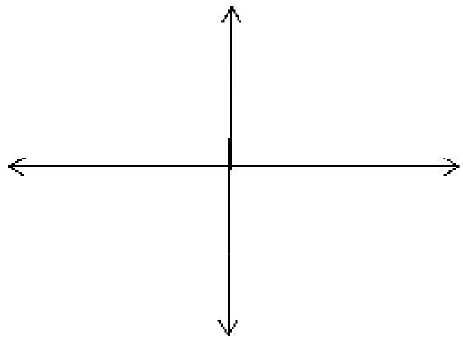
x	y
0	1
1	e

$y = \ln x$

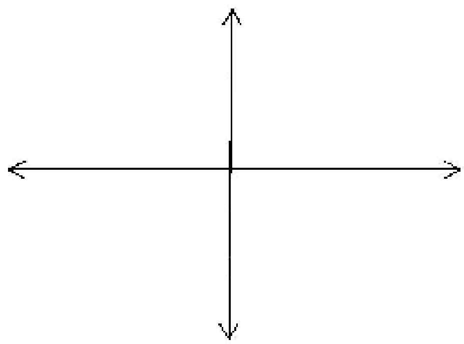
x	y
1	0
e	1



$y = -\ln x + 1$



$y = 2\ln(-x)$



All properties for expanding and condensing logs hold true for expanding and condensing natural logarithms.

Example 5: Expand using natural logs: $\ln\left(\frac{x^3(x-2)^4}{\sqrt{x-5}}\right)$

$\ln x^3 + \ln(x-2)^4 - \ln\sqrt{x-5}$

$3\ln x + 4\ln(x-2) - \frac{1}{2}\ln(x-5)$

$\frac{1}{2}\ln(x+5)$

Example 6: Condense the following: $4\ln x + 3\ln(x^2-4) - \frac{1}{2}\cancel{(x+5)}$

$\ln x^4 + \ln(x^2-4)^3 - \ln(x+5)^{\frac{1}{2}}$

$\ln\left(\frac{x^4(x^2-4)^3}{\sqrt{x+5}}\right)$

—

Example 7: Solve each of the following.

a. $\ln e^{5x} = 7$ $\qquad$ $\ln e = 1$

$5x\ln e = 7$

$5x(1) = 7$

$5x = 7$

$\boxed{x = \frac{7}{5}}$

b. $e^{\ln(2x+1)} = 4x$

c. $e^{3x+7} = 6$

$y = -\ln x + 1$

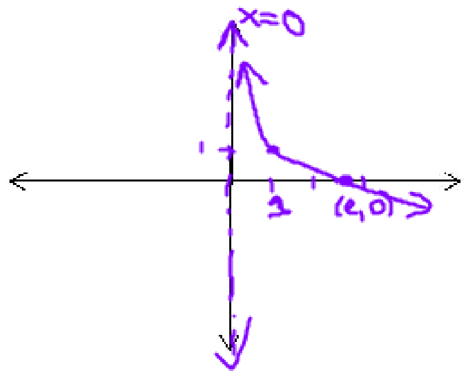
$y = e^x$ $\qquad$ $x = e^y$

x	y
0	1
1	e = 2.718

x	y
1	0
e	1

$y = -\ln x + 1$

x	$-y+1$
1	1
e	0



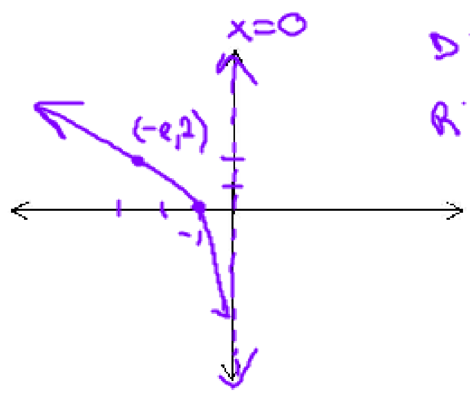
$y = 2\ln(-x)$

$x = e^y$

x	y
1	0
e	1

$y = 2\ln(-x)$

$-x$	$2y$
-1	0
$-e$	2



D: $(-\infty, 0)$

R: $(-\infty, \infty)$

$y \in \mathbb{R}$

All properties for expanding and condensing logs hold true for expanding and condensing natural logarithms.

Example 5: Expand using natural logs: $\ln\left(\frac{x^3(x-2)^4}{\sqrt{x-5}}\right)$

Example 6: Condense the following: $4\ln x + 3\ln(x^2-4) - \frac{1}{2}(x+5)$

—

Example 7: Solve each of the following.

a. $\ln e^{5x} = 7$

b. $e^{\ln(2x+1)} = 4x$

$\ln e = 1$

$\ln\left(e^{\ln(2x+1)}\right) = \ln(4x)$

$\ln(2x+1)(\cancel{\ln e}) = \ln(4X)$

$\ln(2x+1) = \ln(4x)$

$2x+1 = 4X$

$-2x \qquad -2x$

$\frac{1}{2} = \frac{2x}{2}$

$\therefore x = \frac{1}{2}$

$\ln(x^y)$

$y\ln x$

c. $e^{3x+7} = 6$

$\ln e^{3x+7} = \ln 6$

$(3x+7)\ln e = \ln 6$

$3x+7 = \ln 6$

$\frac{3X}{3} = \frac{\ln 6 - 7}{3}$

$X = \frac{\ln 6 - 7}{3}$

$\boxed{\therefore X = -1.736}$

Law of Natural Growth/Decay

$$\boxed{A = Pe^{rt}}$$

A = *final amount*

P = *initial amount*

r = *rate*

t = *time*

Example 8: The population of a bacteria culture is 20 000 and is increasing at a rate of .037 according to natural growth. Find the population after 25 min.

$A=Pe^{rt}$ $\quad P=20000 \quad r=0.037 \quad t=25\text{ min}$

$A=20000e^{(0.037)(25)}$

$A=50437$

Example 9: In a reaction the original concentration of .03 is reduced to .01 in 4 minutes. Find the rate and the concentration in 10 minutes.

$P=0.03 \quad A=0.01 \quad t=4\text{ mins}$

Find r first.

$A=Pe^{rt}$

$\frac{0.01}{0.03}=\frac{0.03e^{r(4)}}{0.03}$

$\frac{1}{3}=e^{4r}$

$\ln(\frac{1}{3})=\ln e^{4r}$

$\frac{\ln(\frac{1}{3})}{4}=\frac{4r\cancel{\ln e}}{4}$

$r=-0.2747$

↳ negative b/c decreasing

Part 2: $t=10$, $r=-0.2747$, $P=0.03$, find A.

$A=0.03e^{(-0.2747)(10)}$

$A=0.00192$

Example 10: The half-life of sodium-24 is 14.9 hours. A hospital buys 40 mg. How much will remain after 48 hours and how long until there is 1 mg remaining?

$P=40 \quad A=20 \quad t=14.9$ hours

$A=Pe^{rt}$

Find r first.

$\frac{20}{40}=\frac{40e^{r(14.9)}}{40}$

$0.5=e^{14.9r}$

$\ln 0.5=\ln e^{14.9r}$

$\frac{\ln 0.5}{14.9}=\frac{14.9r(\ln e)}{14.9}$

$-0.0465=r$

After 48 hours

$A=40e^{(-0.0465)(48)}$

$\boxed{A=4.29\text{ mg}}$

1 mg Remaining

$\frac{1}{40}=\frac{40e^{(-0.0465)t}}{40}$

$0.025=e^{(-0.0465)t}$

$\ln(0.025)=\ln e^{-0.0465t}$

$\frac{\ln(0.025)}{-0.0465}=\frac{-0.0465t(\ln e)}{-0.0465}$

$\boxed{79.33\text{ hours}=t}$