Chapter 6 Trigonometry

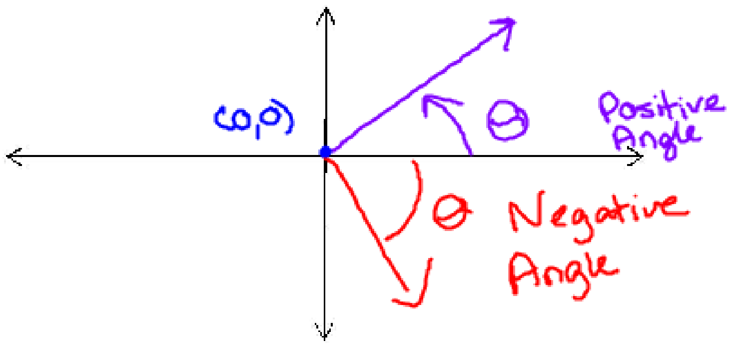
6.1 Trigonometric Ratios for Any Angle in Standard Position

An angle is said to be in **standard position** if it has its centre at the origin and its initial arm is along the positive x-axis.

A **positive angle** will be measured counter-clockwise from the positive x-axis.

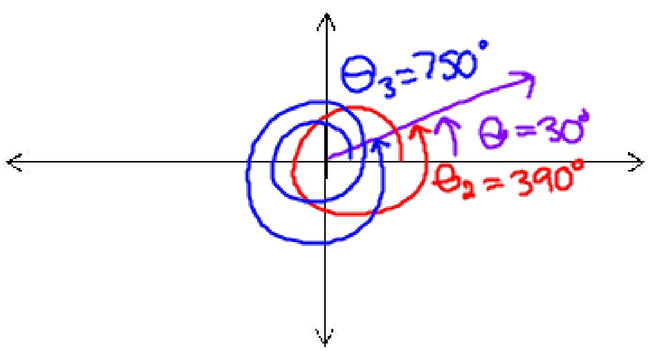
A **negative angle** will be measured clockwise from the positive x-axis.

Label the origin, the positive x-axis, a positive angle and a negative angle.

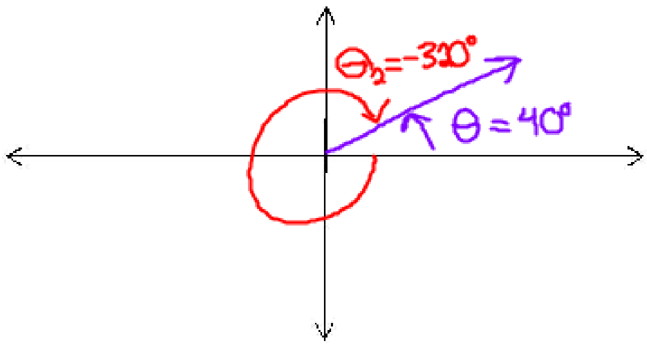


Angles in standard position with the same terminal arm are called **coterminal angles**.

30° and 390° and 750° are *coterminal angles*.

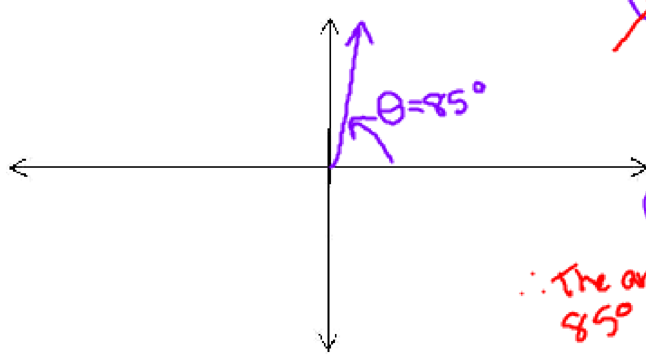


40^{0} and -320^{0} and -680° are coterminal.



Example 1:

a. Determine the measures of all angles in standard position between -800° and 800° that are coterminal with an angle of 85° in standard position. Sketch the angles.
b. Write an expression for the measures of all the angles that are coterminal with an angle of 85° in standard position.



$\theta_2 = 85 + 360 = 445^\circ$

$\cancel{\theta_3} = 445 + 360 = 805^\circ$

↳ Too big.

$\theta_4 = 85 - 360 = -275^\circ$

$\theta_5 = -275 - 360 = -635^\circ$

∴ The angles coterminal with 85° b/w -800° & 800° are 445°, -275° & -635°.

Suppose a terminal point $P(x, y)$, in any quadrant, lies 1 unit from the origin O. As the terminal arm OP rotates, P traces a circle.

The equation of a circle is $x^2 + y^2 = r^2$.

A circle with a centre at the origin and radius 1 unit, has equation $\boxed{x^2 + y^2 = 1}$. This is the equation of the **unit circle**.

radius=1

SOH CAH TOA

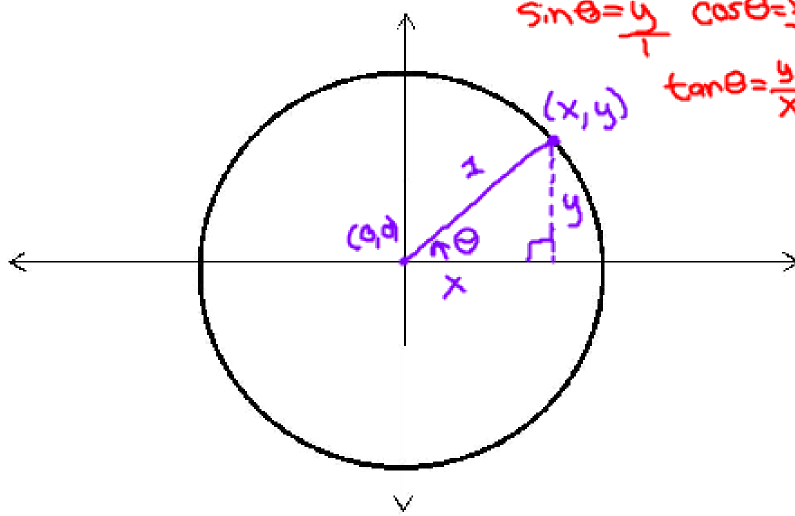
$\sin\theta = \frac{y}{1}$ $\cos\theta = \frac{x}{1}$

$\tan\theta = \frac{y}{x}$

Recall:

$\sin\theta = \frac{y}{r}$ $\quad\cos\theta = \frac{x}{r}$ $\quad\tan\theta = \frac{y}{x}$

For a unit circle these become:



$\sin\theta = y$ $\quad\cos\theta = x$ $\quad\tan\theta = \frac{y}{x}$

Reciprocal Trigonometric Ratios

cosecant
csc

$\csc\theta = \frac{1}{\sin\theta}$

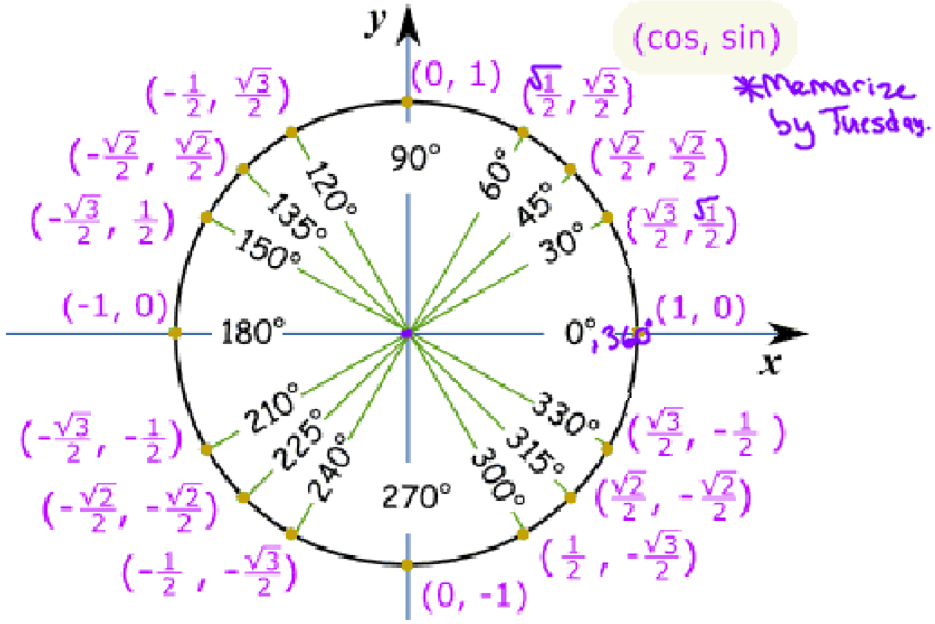
secant
sec

$\sec\theta = \frac{1}{\cos\theta}$

cotangent
cot

$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$

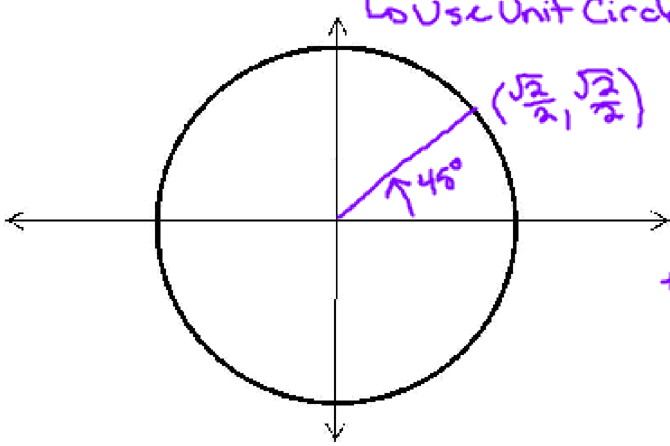
Recall the exact values of the unit circle.



Example 2:

a. Determine the exact values of the six trigonometric ratios for 45°.

→ Use Unit Circle!



$x = \cos\theta$

$y = \sin\theta$

$\cos\theta = \frac{\sqrt{2}}{2}$

$\sin\theta = \frac{\sqrt{2}}{2}$

$\tan\theta = \frac{y}{x} = \frac{\left(\frac{\sqrt{2}}{2}\right)}{\left(\frac{\sqrt{2}}{2}\right)} = 1$

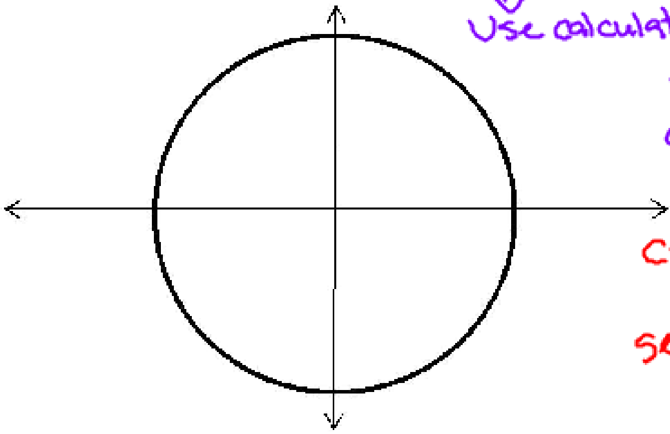
$\sec\theta = \frac{1}{\cos\theta} = \frac{2}{\sqrt{2}}$

$\csc\theta = \frac{1}{\sin\theta} = \frac{\sqrt{2}}{2}$

$\cot\theta = \frac{1}{\tan\theta} = 1$

b. Determine the approximate values of the six trigonometric ratios for -416°, to the nearest thousandth.

Use calculator. → in Degrees!



$\sin(-416°) = -0.829$

$\cos(-416°) = 0.559$

$\tan(-416°) = -1.483$

$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{-0.829} = -1.206$

$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{0.559} = 1.789$

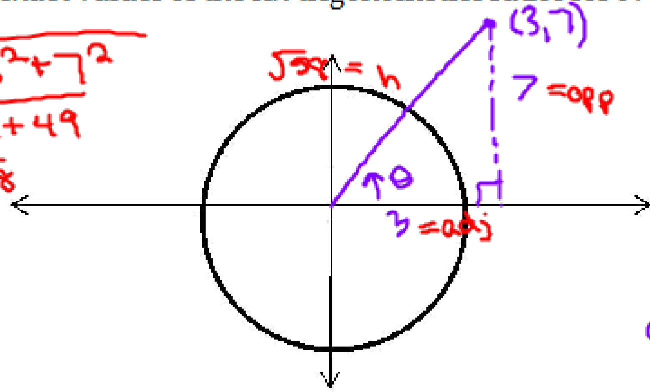
$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{-1.483} = -0.674$

Example 3: $P(3,7)$ is a terminal point of angle θ in standard position. Determine the exact values of the six trigonometric ratios for θ.

$h = \sqrt{3^2 + 7^2}$

$h = \sqrt{9 + 49}$

$h = \sqrt{58}$



$\sin\theta = \frac{7}{\sqrt{58}}$

$\cos\theta = \frac{3}{\sqrt{58}}$

$\tan\theta = \frac{7}{3}$

$\csc\theta = \frac{\sqrt{58}}{7}$

$\sec\theta = \frac{\sqrt{58}}{3}$

$\cot\theta = \frac{3}{7}$

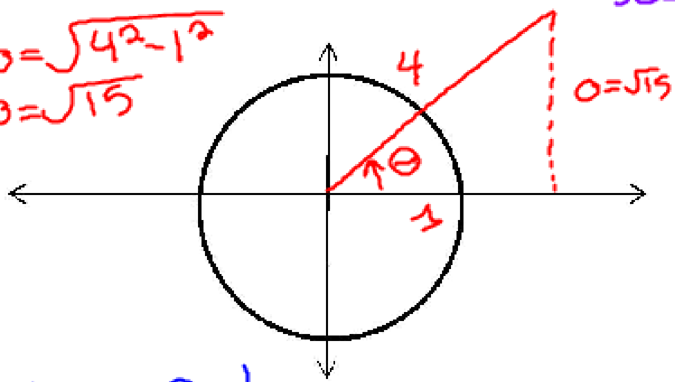
Example 4: Suppose $\sec\theta = 4$.

a. Determine the exact values of the other trigonometric ratios for $0° \le \theta \le 180°$.
b. To the nearest degree, determine possible values of θ in the domain $-360° \le \theta \le 360°$.

$\sec\theta = 4 \quad \cos\theta = \frac{1}{4} = \frac{adj}{hyp}$

$o = \sqrt{4^2 - 1^2}$

$o = \sqrt{15}$



$\sin\theta = \frac{O}{H} = \frac{\sqrt{15}}{4}$

$\csc\theta = \frac{4}{\sqrt{15}}$

$\tan\theta = \frac{O}{A} = \frac{\sqrt{15}}{1}$

$\cot\theta = \frac{1}{\sqrt{15}}$

b) $\cos\theta = \frac{1}{4}$

$\theta = \cos^{-1}\left(\frac{1}{4}\right)$

$\theta = 75.5°$

$\theta = 76°$

$\theta = 180° - \theta_r$ II | I $\theta = \theta_r$

sin θ is positive ← S | A → All trig ratios positive

tan θ is positive ← T | C → cos θ is positive

III | IV

$\theta = 180° + \theta_r \quad \theta = 360° - \theta_r$

Cos θ is positive

∴ θ is in Q I or Q IV

In Q I, $\boxed{\theta = 76°}$

In Q IV, $360 - 76° = \theta$

$\boxed{\theta = 284°}$

6.2 Angles in Standard Position and Arc Length

Radians

Another unit for measuring angles is the **radian**.

In a circle with radius r: A central angle of 1 radian is subtended by an arc with length r.

A central angle of θ radians is subtended by an arc with length $r\theta$; conversely, the length of an arc, a, that subtends a central angle of θ radians is: $\boxed{a = r\theta.}$

1 radian = $\frac{180°}{\pi}$

$1° = \frac{\pi}{180°}$ radians

$\boxed{\pi \text{ radians} = 180°}$

$\boxed{360° = 2\pi \text{ radians}}$

Example 1: Convert the following degrees to radians and express your answer in terms of π and in decimal form using the ratio:

$$\frac{\pi}{180°} = \frac{\text{radians}}{\text{degrees}} \quad *$$

a. 255°

$$\frac{\pi}{180°} = \frac{\text{radians}}{255°}$$

$$\text{radians} = \frac{255\pi}{180} = \frac{51\pi}{36} = \frac{17\pi}{12}$$

b. 300°

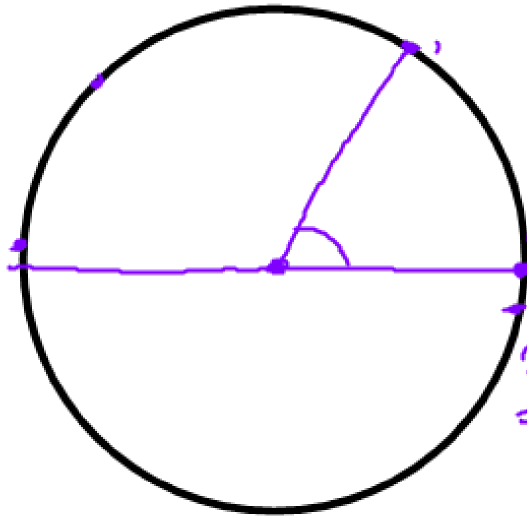
$$\frac{\pi}{180°} = \frac{\text{radians}}{300°}$$

$$\text{radians} = \frac{300\pi}{180} = \frac{30\pi}{18} = \frac{10\pi}{6} = \frac{5\pi}{3}$$

c. $-75°$

$$\frac{\pi}{180°} = \frac{\text{radians}}{-75°}$$

$$\text{radians} = \frac{-75°\pi}{180°} = \frac{-15\pi}{36} = \frac{-5\pi}{12}$$



$3.14 = \pi$

π radians $= 180°$

2π radians $= 360°$

Example 2: Convert the following radians to degrees and express your answer in decimal form using the fact that $\pi = 180°$ and the ratio:

$$\frac{\pi}{180°} = \frac{\text{radians}}{\text{degrees}}$$

a. $\frac{\pi}{7}$ radians

$$\frac{\pi}{180°} = \frac{\frac{\pi}{7}}{\text{degrees}}$$

$$\text{degrees} = \frac{180\left(\frac{\pi}{7}\right)}{\pi} = \frac{\frac{180\pi}{7}}{\pi} = \frac{180\pi}{7\pi} = \frac{180°}{7}$$

b. $\frac{-2\pi}{9}$ radians

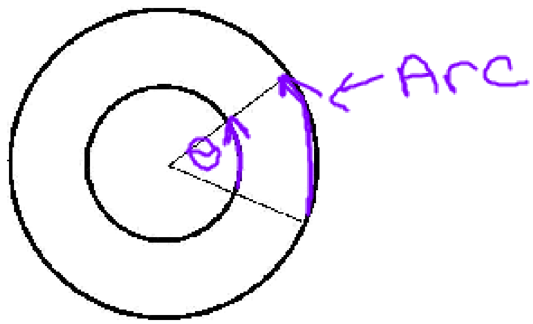
$$\frac{\pi}{180°} = \frac{-\frac{2\pi}{9}}{\text{degrees}}$$

$$\text{degrees} = \frac{180\left(-\frac{2\pi}{9}\right)}{\pi} = \frac{-\frac{360\pi}{9}}{\pi} = \frac{-360\pi}{9\pi} = -\frac{360}{9} = -40°$$

c. 2.5 radians

Arc Length

All arcs that subtend a given angle, θ, have the same central angle, but they will have a different arc length depending on the radius of the circle. The arc length is proportional to the radius.



The formula which connects an arc length, the central angle and the radius is given by $\boxed{a = r\theta.}$ The angle θ must be measured in $\boxed{\text{radians}}$ (a unit of measure in which angles are represented as a fraction of π) and the units for the arc length and the radius must be the same.

arc=(radius)(angle)

Example 1: Determine the unknown quantity.

a. $r = 9.2, \theta = \frac{\pi}{5}, a = ?$

$a = r\theta$

$a = 9.2\left(\frac{\pi}{5}\right)$

$a = 5.78$

b. $r = 4.8, \theta = 115°, a = ?$

Convert 115° to radians.

$\frac{\pi}{180} = \frac{\text{radians}}{\text{degrees}}$

$\theta = \frac{115\pi}{180}$

$a = (4.8)\left(\frac{115\pi}{180}\right)$

$a = 9.63$

c. $r = 5.8, \theta = ?, a = 235$

$a = r\theta$

$\frac{235}{5.8} = \frac{5.8\,\theta}{5.8}$

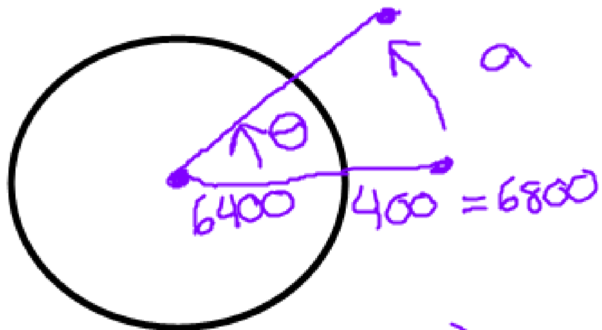
$\theta = 40.5$ radians

6.3 Radian Measure and Graphing Trig Functions

Example 1: An approximate model of the motion of the international space station is that it travels at a speed of 27 600 km/h in a circular orbit at an altitude of 400 km. The radius of Earth is approximately 6400 km.

Visualize a line segment joining the space station to the centre of Earth. To the nearest tenth of a radian, through which angle will the segment have rotated after 40 min?

$a = \theta r$



$\frac{18400}{6800} = \frac{\theta(6800)}{6800}$

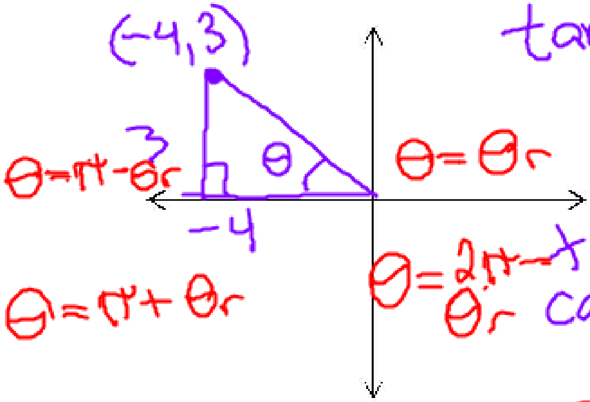
$\boxed{\theta = 2.7 \text{ radians}}$

$\frac{40}{60}(27600) = 18400\text{km}$

Example 2:

a. $P(-4, 3)$ is a terminal point of angle θ in standard position. To the nearest tenth of a radian, determine possible values of θ in the domain of $\boxed{-2\pi \le \theta \le 2\pi}$.

radians



$\tan\theta_r = \left(\frac{3}{4}\right)$

Use positive ratio when typing into calc.

$\theta_r = \tan^{-1}\left(\frac{3}{4}\right)$

$\theta_r = 0.6$

+ Q2

$\theta = \pi - 0.6$

$\boxed{\theta = 2.5}$

− Q2

$\theta = 2.5 - 2\pi$

$\boxed{\theta = -3.8}$

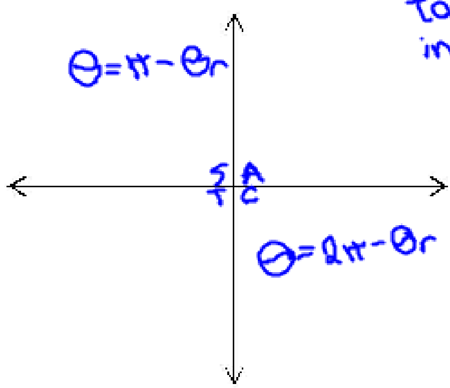
b. Given $\cot\theta = -2$; to the nearest tenth of a radian, determine the values of θ in the domain $\boxed{-2\pi \le \theta \le 2\pi.}$

$\tan\theta = -\frac{1}{2}$

tan is negative in II, IV

$\theta_r = \tan^{-1}\left(\frac{1}{2}\right)$

$\theta_r = 0.5$



+ Q II: $\theta = \pi - 0.5$, $\boxed{\theta = 2.6}$

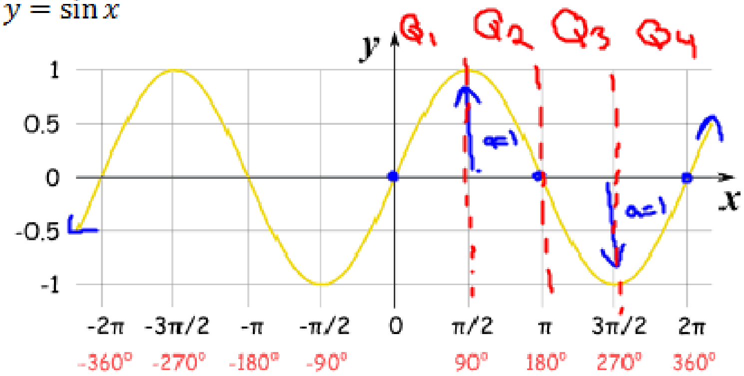
+ Q IV: $\theta = 2\pi - 0.5$, $\boxed{\theta = 5.8}$

− Q II: $\theta = 2.6 - 2\pi$, $\boxed{\theta = -3.7}$

− Q IV: $\theta = 5.8 - 2\pi$, $\boxed{\theta = -0.5}$

Example 3: Graphs of Trig Functions

a. $y = \sin x$



Properties:

Domain $(-\infty, \infty)$

Range $[-1, 1]$

Period 2π

Amplitude 1

Zeros $k\pi$, $k \in$ integer

~~Equations of Asymptotes~~

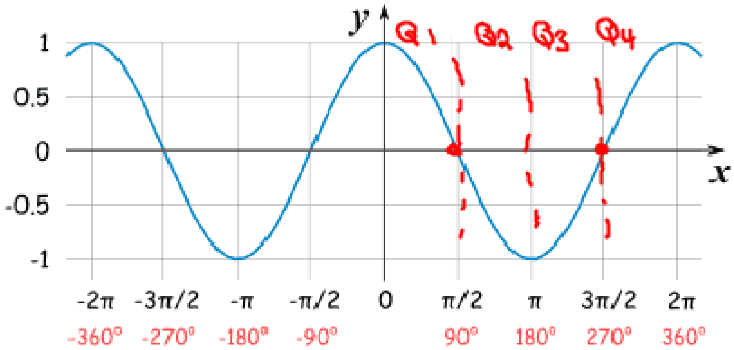
Positive Q1 & Q2

Negative Q3 & Q4

Increasing Q1 & Q4

Decreasing Q2 & Q3

b. $y = \cos x$



Domain $(-\infty, \infty)$

Range $[-1, 1]$

Period 2π

Amplitude 1

Zeros $\frac{\pi}{2} + k\pi$, $k \in$ integer

~~Equations of Asymptotes~~

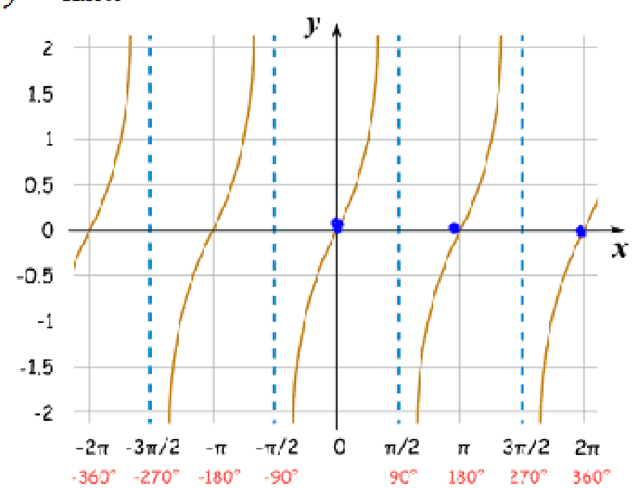
Positive Q_1 & Q_4

Negative Q_2 & Q_3

Increasing Q_3 & Q_4

Decreasing Q_1 & Q_2

c. $y = \tan x$



Domain $x \in \mathbb{R}$, $x \neq \frac{\pi}{2} + k\pi$, $k \in$ integer

Range $(-\infty, \infty)$

Period π

~~Amplitude~~ No

Zeros $k\pi$, $k \in$ integers

Equations of Asymptotes $x = \frac{\pi}{2} + k\pi$

Positive Q_1 & Q_3

Negative Q_2 & Q_4

Increasing always

Decreasing never

6.5 Trigonometric Functions

A function that repeats its values in regular intervals over its domain is a **periodic function**.

The length of each interval, or cycle, measured along the horizontal axis is called the **period** of the function.

sine wave

Functions whose graphs have the same shape as $y = \sin x$ or $y = \cos x$ are **sinusoidal functions**.

The **amplitude** of a sinusoidal function is the distance of a maximum or minimum point from the centre line or mid-point line.

$f = \sin$

$f = \cos$

$y = af[b(x-h)]+k$

Recall the transformations involved in the graphing of $y - k = af[b(x - h)]$.

The **amplitude** of $y = a\sin x$ and $y = a\cos x$ is $|a|$.

Example 1: Determine the amplitude of the graph of each function.

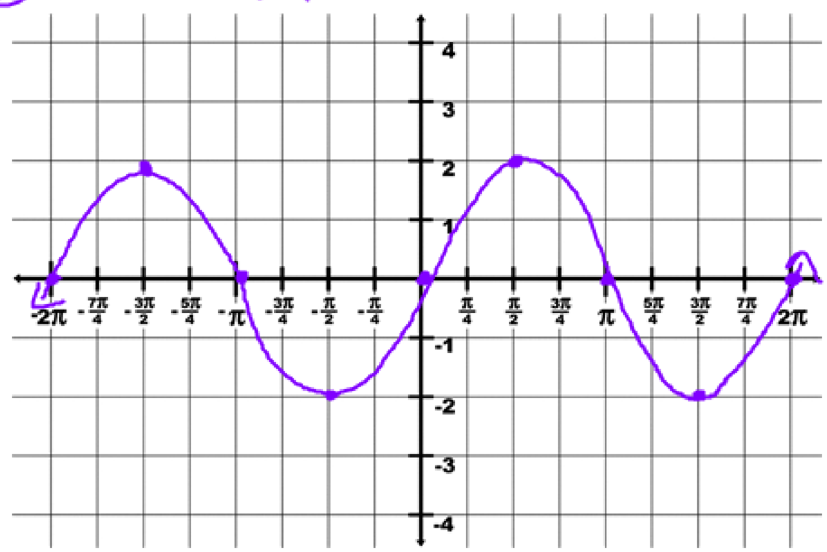
a. $y = \frac{2}{3}\sin x$ Amplitude $= \frac{2}{3}$

b. $y = -4\cos x$ Amplitude $= 4$

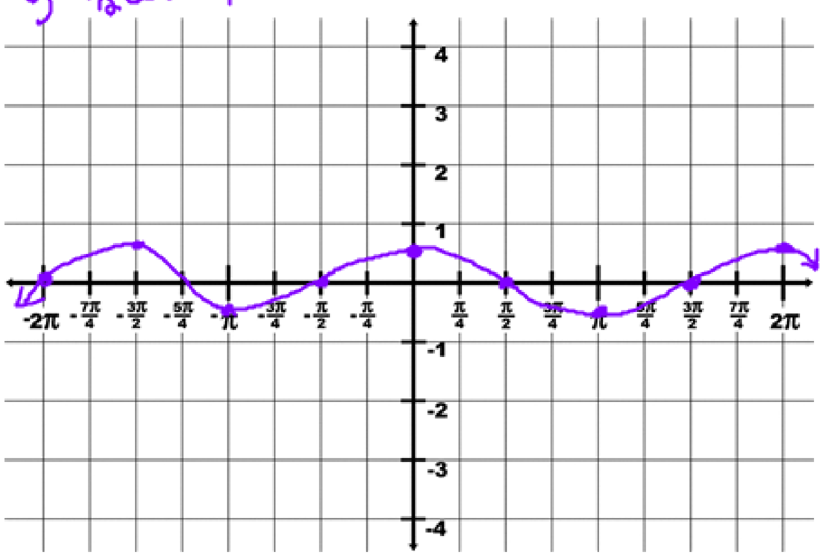
Example 2: Sketch $y = 2\sin x$ and $y = \frac{1}{2}\cos x$.

a

Amp = 2



$y = \frac{1}{2}\cos(x)$



The **period** of $y = \sin bx$ and $y = \cos bx$ is $\boxed{\frac{2\pi}{b}}$, $b > 0$.

$\frac{2\pi}{b} = \frac{360}{b}$

~~The period of $y = \tan bx$ is $\frac{\pi}{b}$, $b > 0$.~~

The interval (scale) for sketching the new function will be $\frac{\text{new period}}{4}$.

Example 3: Determine the period of each function.

a. $y = \cos 6x$ Period $= \frac{2\pi}{6} = \frac{\pi}{3}$

b. $y = \tan\left(\frac{2}{3}x\right)$ Period $= \frac{2\pi}{\frac{2}{3}} = 2\pi\left(\frac{3}{2}\right) = \frac{6\pi}{2} = 3\pi$

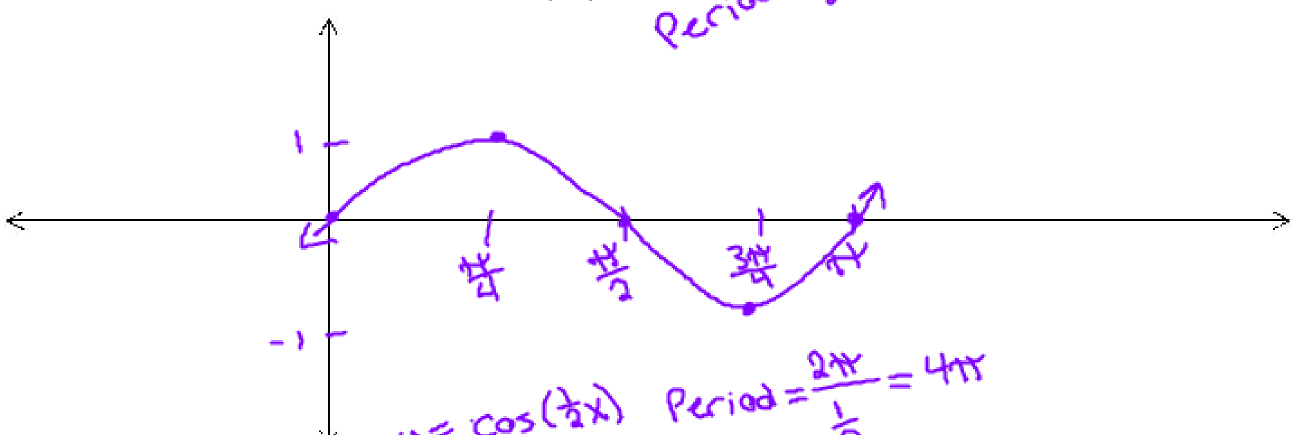
$b = \frac{1}{7}$

c. $y = \sin\left(\frac{x}{7}\right)$ Period $= \frac{2\pi}{\frac{1}{7}} = 2\pi\left(\frac{7}{1}\right) = 14\pi$

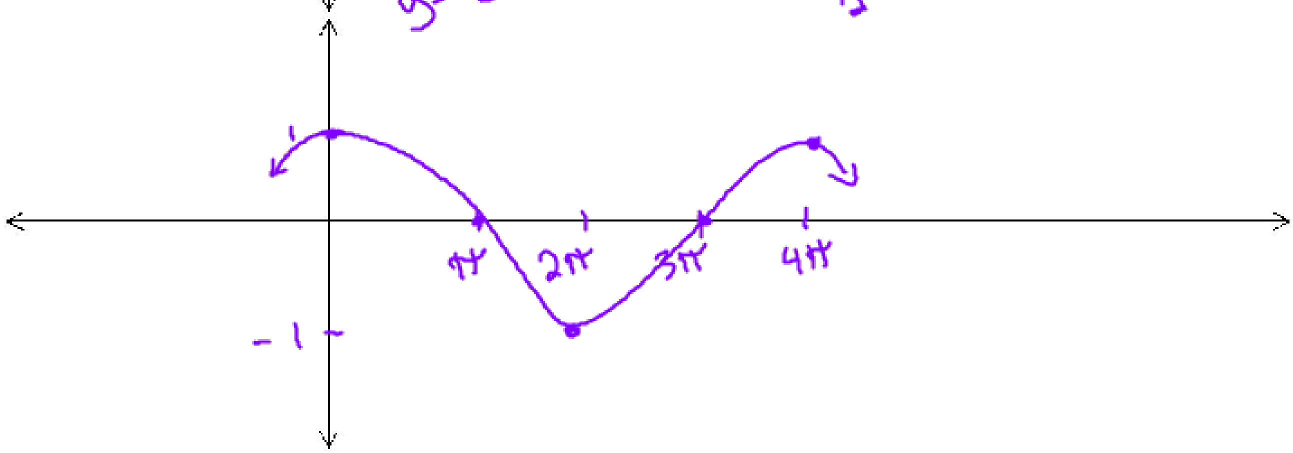
Example 4: Sketch $y = \sin 2x$ and $y = \cos\left(\frac{1}{2}x\right)$.

$y = \sin 2x$

Period $= \frac{2\pi}{2} = \pi$



$y = \cos\left(\frac{1}{2}x\right)$ Period $= \frac{2\pi}{\frac{1}{2}} = 4\pi$



The **phase shift** is the horizontal distance c that the curves of $y = \sin x$ and $y = \cos x$ have been translated and is shown as $y = \sin(x - c)$ and $y = \cos(x - c)$.

Example 5:

a. Determine the phase shift of the function $y = \sin\left(x + \frac{\pi}{4}\right)$.

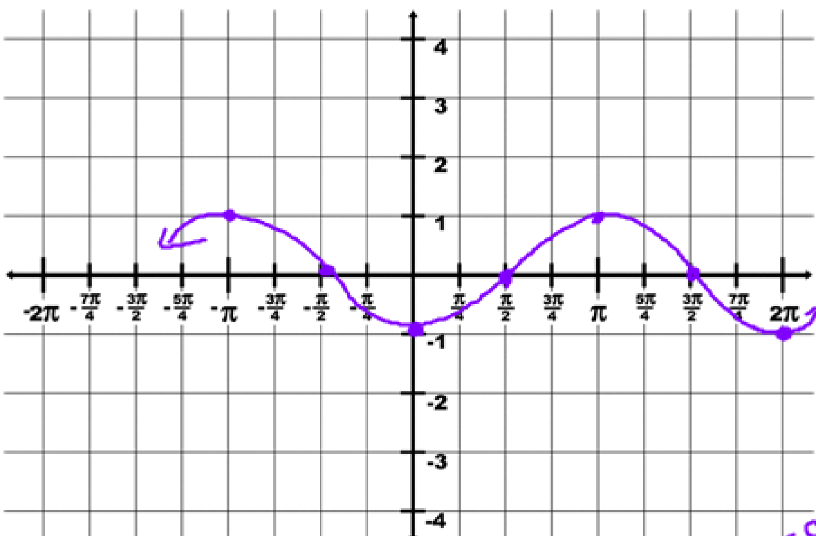
sin(x) is shifted left $\frac{\pi}{4}$.

a) Determine the phase shift of the function $y = \cos\left(x - \frac{\pi}{6}\right)$.

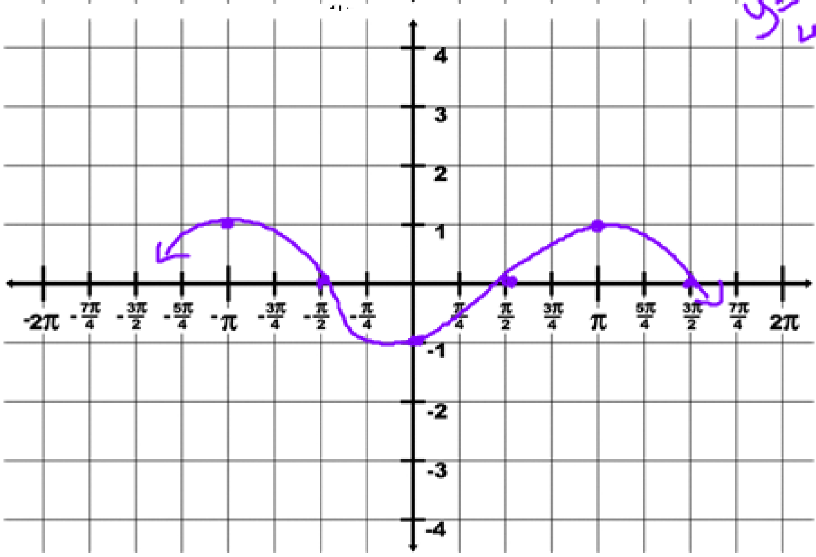
cos(x) is shifted right $\frac{\pi}{6}$.

Right $\frac{\pi}{2}$

Example 6: Sketch $y = \sin\left(x - \frac{\pi}{2}\right)$ and $y = \cos(x + \pi)$.



$y = \cos(x + \pi)$
↳ Left π



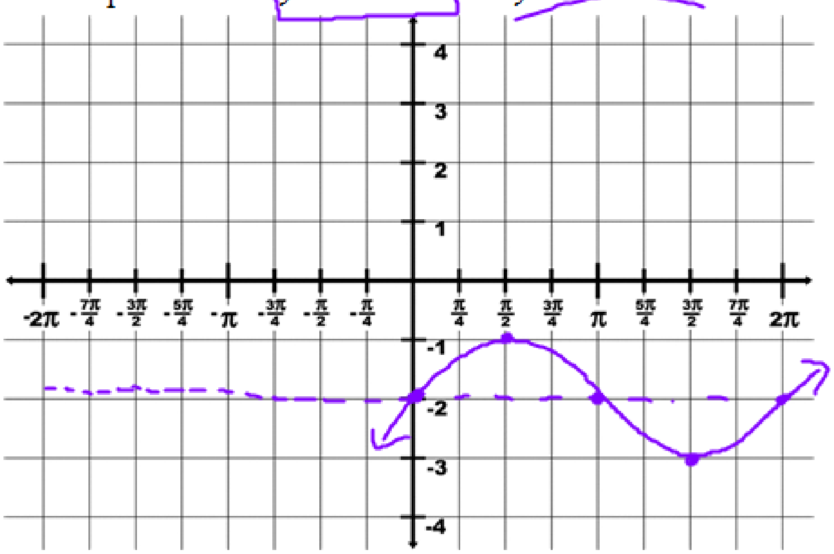
The **middle** of a sinusoidal graph or the vertical translation is the value of d in $y = \sin x + d$ or $y = \cos x + d$.

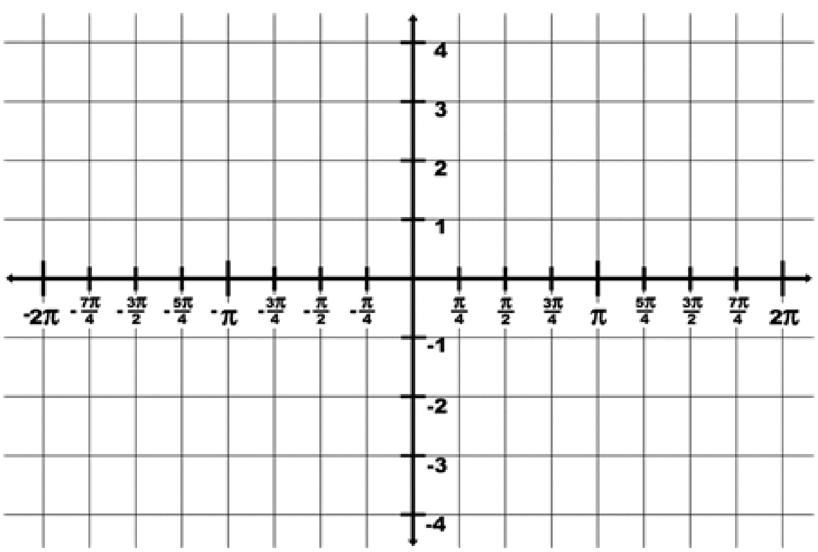
Example 7: State the middle value of each of the following.

a. $y = \sin x - 1$ Middle is $y=-1$

b. $y = \cos x + 3$ Middle is $y=3$.

Example 8: Sketch $\boxed{y = \sin x - 2}$ and ~~$y = \cos x + 1$~~.





Summary

For $y = a\sin[b(x - c)]$ and $y = a\cos[b(x - c)] + d$:

- a will affect a vertical stretch or compression, a is the amplitude, and if $a < 0$ there will be a reflection in the x-axis
- b will affect a horizontal stretch or compression
  - the new period of $y = \sin bx$ and $y = \cos bx$ is $\frac{2\pi}{b}$, $b > 0$
  - the new period of $y = \tan bx$ is $\frac{\pi}{b}$, $b > 0$.
  - The interval will be $\frac{\text{new period}}{4}$.
- c will affect a horizontal translation.
- d will affect a vertical translation and is the new middle.

6.6 Combining Transformations of Sinusoidal Functions

Consider the sinusoidal functions :

$$y = a\sin[b(x-c)] \text{ and } y = a\cos[b(x-c)] + d$$

Example 1: Complete the chart.

	$y=\sin 2x$	$y=\frac{1}{2}\sin 2x$	$y=\frac{1}{2}\sin 2\left(x-\frac{\pi}{4}\right)$	$y=\frac{1}{2}\sin 2\left(x-\frac{\pi}{4}\right)-3$
Period $\frac{2\pi}{b}$	$\frac{2\pi}{2}=\pi$	π	π	π
Amplitude	1	$1/2$	$1/2$	$1/2$
Domain	$(-\infty,\infty)$ →			
Range	$[-1,1]$	$[-\frac{1}{2},\frac{1}{2}]$	$[-\frac{1}{2},\frac{1}{2}]$	$[3.5,-2.5]$
Phase Shift	—	—	$\frac{\pi}{4}$ right	$\frac{\pi}{4}$ right
Zeros $k\pi$, $k\in I$	$\frac{k\pi}{2}, k\in I$	$\frac{k\pi}{2}, k\in I$	$\frac{\pi}{4}+\frac{k\pi}{2}, k\in I$	None

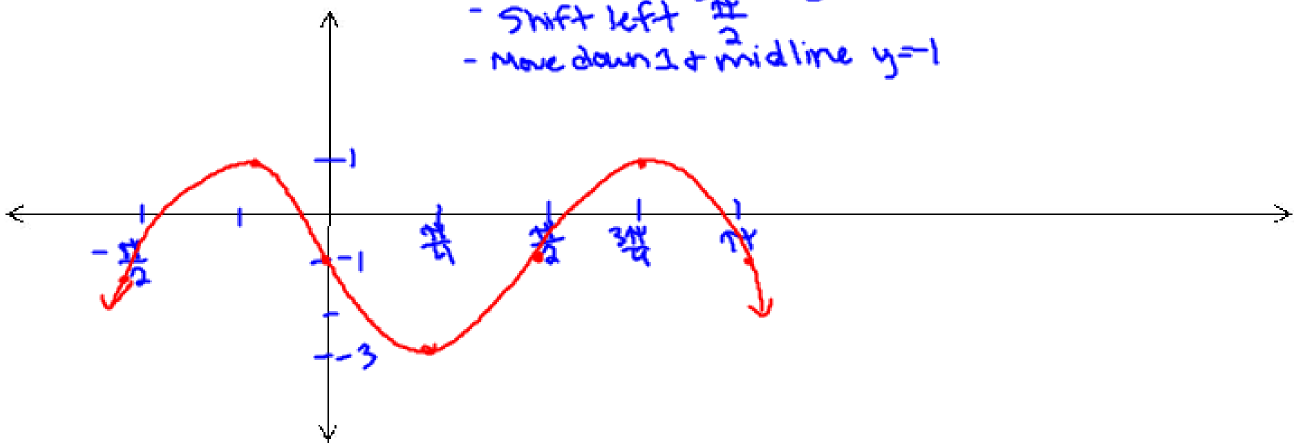
Steps to graphing a sinusoidal function.

1. Draw the main curve.
2. Account for the b value and calculate the new interval. Label the x-axis.
3. Account for the a and d values. Label the y-axis.
4. Account for the c value.

Example 2: Graph and explain how the given graph is related to the main graph.

a. $y = 2\sin 2\left(x + \frac{\pi}{2}\right) - 1$

- Amplitude $= 2$
- Period $= \frac{2\pi}{b} = \frac{2\pi}{2} = \pi$
- Shift left $\frac{\pi}{2}$
- Move down 1 & midline $y = -1$



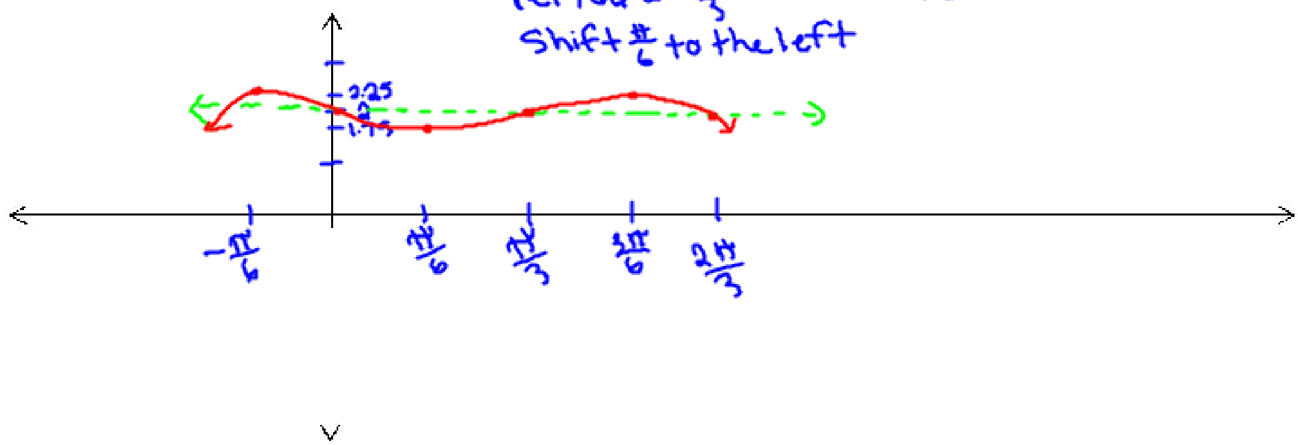
b. $y = \frac{1}{4}\cos 3\left(x + \frac{\pi}{6}\right) + 2$

Amp $= \frac{1}{4}$

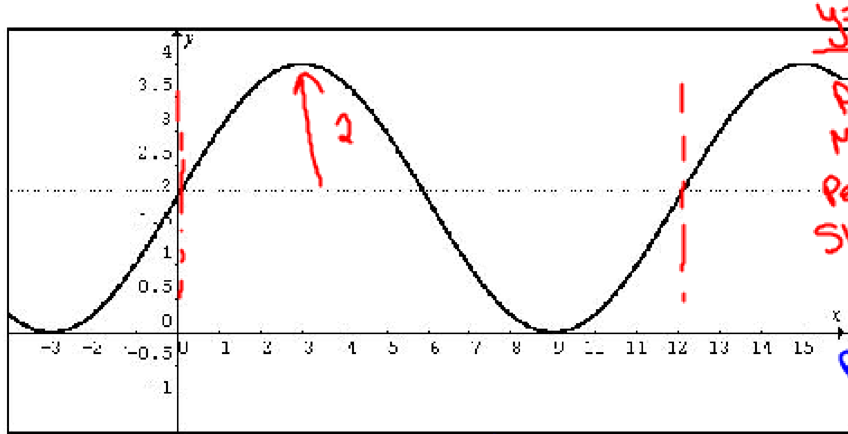
Period $= \frac{2\pi}{3}$

Shift $\frac{\pi}{6}$ to the left

Shift 2 up, midline $= 2$



Example 3: Write the sinusoidal equation for the following graph.



$y = \sin x$

Amp $= 2 = a$

Midline $= 2 = d$

Period $= 12$

Shift $= 0 = c$

$y = a\sin b(x-c) + d$

Period $= \frac{2\pi}{b}$

$12 = \frac{2\pi}{b}$

$b = \frac{2\pi}{12} = \frac{\pi}{6}$

$\therefore y = 2\sin\left(\frac{\pi}{6}(x)\right) + 2$

$y = 2\cos\left(\frac{\pi}{6}(x-3)\right) + 2$

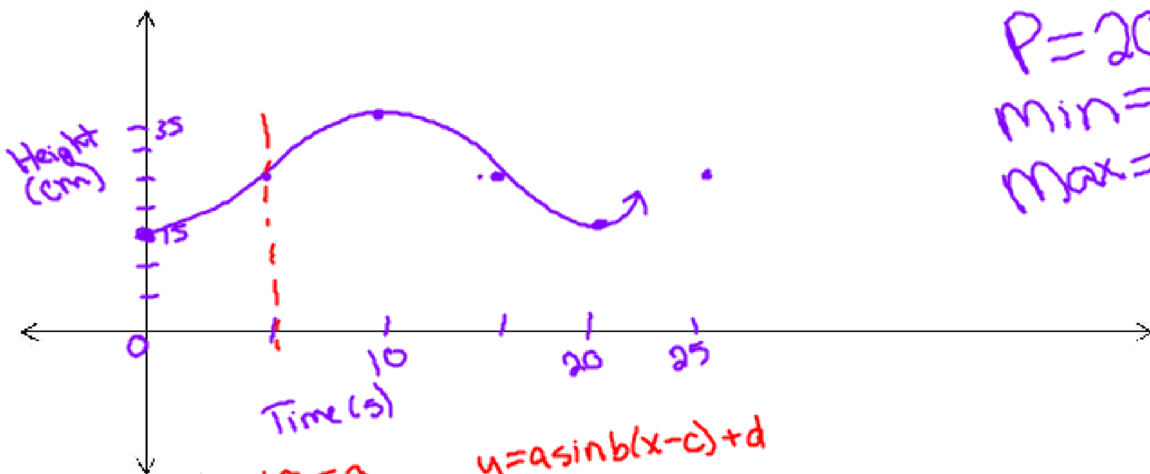
6.7 Applications of Sinusoidal Functions

The horizontal axis of a sinusoidal graph is often labelled in terms of π. When sinusoidal graphs are used in applications, the horizontal axis usually represents time and the axis can be labelled in whole numbers. There are many real world events that can be modelled by a sinusoidal graph with a maximum and minimum value. An oscillating mass on a spring, a Ferris wheel and the flow of the tides are a few examples.

Example 1: A piston moves vertically in a cylinder starting from its minimum height. Every 20 s, the piston repeats its cycle from a minimum height of 15 cm to a maximum height of 35 cm back to a minimum height of 15 cm.

a. Determine a sinusoidal function that models the height, h centimetres, of the piston at time t seconds after it begins moving.
b. Graph the function, then estimate the height of the piston 26 s after it begins moving.

$P = 20$ s
min $= 15$ cm
Max $= 35$ cm



$A = 10 = a$

Midline $= 25 = d$

Period $= \frac{2\pi}{b} = 20$

$\frac{2\pi}{20} = b$

$\frac{\pi}{10} = b$

$c = 5$

$y = a \sin b(x - c) + d$

$\therefore y = 10 \sin \frac{\pi}{10}(x - 5) + 25$

b) After 26 s, the height is approximately 26 cm.

Example 2: Given the following sinusoidal equation: $P(t) = 3000\sin\left[\frac{\pi}{10}(t - 2010)\right] + 10\,000$, determine the maximum value of $P(t)$ and a value of t at which this maximum occurs. (3 marks)

Maximum depends on amplitude & midline

$a = 3000$ $\quad d = 10000$

∴ The max is $10000 + 3000 = 13000$.

$P(t) = 13000$

$$13000 = 3000\sin\left[\tfrac{\pi}{10}(t-2010)\right] + 10000$$

$$-10000 \qquad -10000$$

$$\frac{3000}{3000} = \frac{3000\sin\left[\frac{\pi}{10}(t-2010)\right]}{3000}$$

We know $\sin\frac{\pi}{2} = 1$

$1 = \sin\frac{\pi}{2}$

$$1 = \sin\left[\tfrac{\pi}{10}(t-2010)\right]$$

$$\therefore \frac{\frac{\pi}{2}}{\frac{\pi}{10}} = \frac{\frac{\pi}{10}(t-2010)}{\frac{\pi}{10}}$$

$$\frac{10\pi}{2\pi} = t - 2010$$

$$5 = t - 2010$$

$$\boxed{\therefore t = 2015}$$