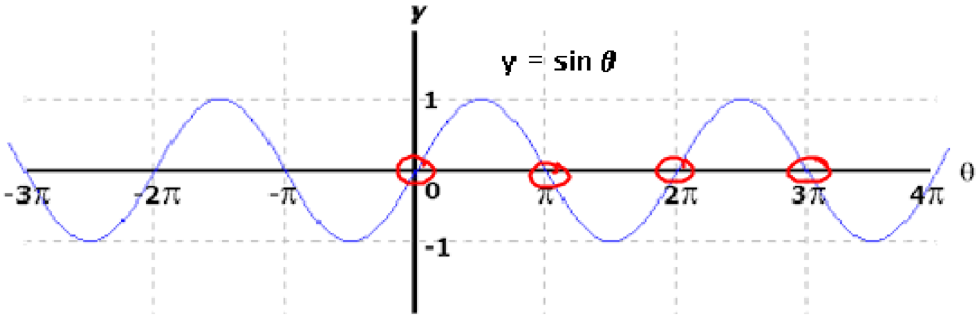
Chapter 7 Trigonometric Equations and Identities

7.1 Solving Trigonometric Equations Graphically

Consider the standard sine graph.



How does this graph relate to the equation $\boxed{\sin\theta = 0?}$

If you were to solve the equation $\sin\theta = 0$, the solutions would be $\theta = 0, \pi, 2\pi, 3\pi$ ... and continue on indefinitely. The general solution would be $\boxed{\theta = \pi n, n \in I.}$

↳ Multiples of π

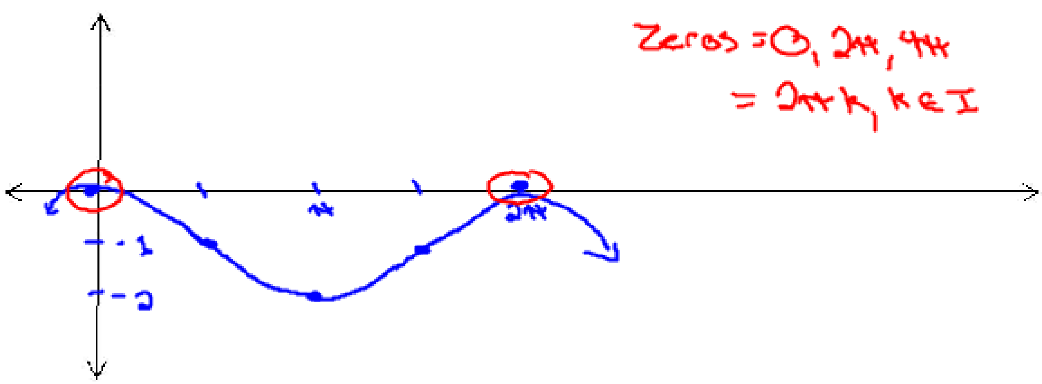
These solutions are the same as the zeros, or x-intercepts, on the graph of $y = \sin\theta$.

Example 1

a. Determine the zeros of the function $\boldsymbol{y = \cos\theta - 1}$.

↳ Moves down 1.

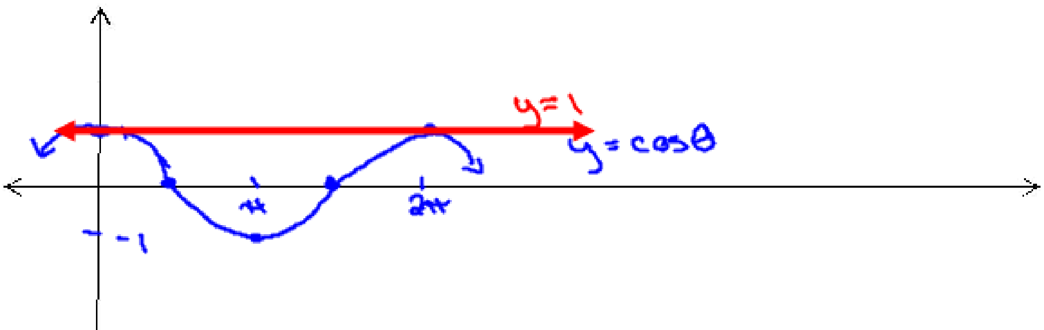
Zeros = 0, 2π, 4π
= 2πk, k ∈ I



$\cos\theta - 1 = 0$
$\cos\theta - 1 = y$

b. Find the general solution to the equation $\cos\theta = 1$. Explain how the two solutions are related.

Graph: $y = \cos\theta$ & $y = 1$



∴ The graphs intersect at 0 & 2π
$x = 2\pi k, k \in I$

Example 2:

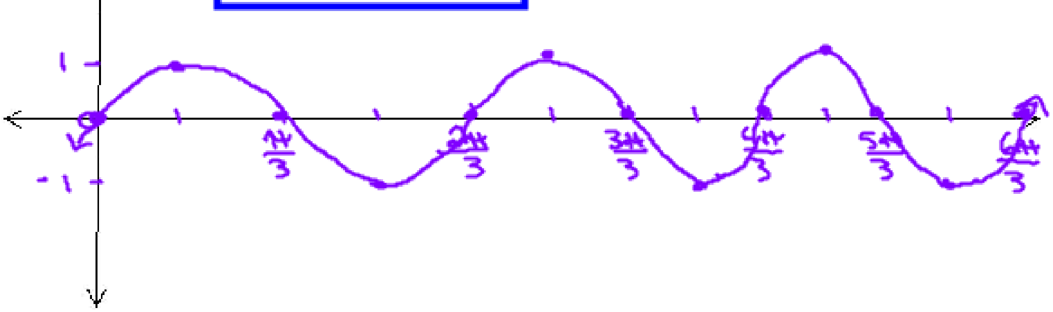
a. Solve $\sin 3x = 0$ over the domain $0 \le x < 2\pi$. Give the roots to the nearest hundredth.

Period $= \frac{2\pi}{3}$

b. Determine the general solution to the equation.

x ints $= 0, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{3}, \frac{4\pi}{3}$

$\boxed{x = \frac{\pi}{3}k, k \in I}$



7.2 Solving Trigonometric Equations Algebraically

Example 1: Solve each of the following trigonometric equations using exact values on the interval $[0, 2\pi]$. → radians

S A
T C

No decimals
∴ Use unit circle

a. $\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

b. $\tan\theta = \sqrt{3}$

Q1, Q3

$\tan\theta = \frac{y}{x}$

$\theta = \frac{\pi}{3}, \frac{4\pi}{3}$

Example 2: Solve each of the following trigonometric equations using a calculator on the interval $[0, 2\pi]$. ← radians

a. $\sin\theta = 0.81$

$\theta = \sin^{-1}(0.81)$

$\theta = 0.94$

Sin is positive.
∴ θ = Q1 & Q2

Q1
$\theta = 0.94$

Q2
$\theta = \pi - 0.94$
$\theta = 2.19$

b. $\tan\theta = -1.45$

$\theta = \tan^{-1}(1.45)$
$\theta_r = 0.97$

tan is negative in Q2 & Q4

Q2
$\theta = \pi - 0.97$
$\theta = 2.17$

Q4
$\theta = 2\pi - 0.97$
$\theta = 5.32$

Example 3: Solve the following trigonometric equation using exact values on the interval $[0°, 360°]$.

$$\tan\theta = \frac{1}{\sqrt{3}}$$

$\theta = 30°, 210°$

Example 4: Solve the following trigonometric equation using a calculator on the interval [0°, 360°].

degrees

$$\sin\theta = -0.38$$

$\theta = \sin^{-1}(0.38)$

$\theta_r = 22.3°$

sin is negative in Q3 & Q4

Q3

$\theta = 180 + 22.3 = 202.3°$

Q4

$\theta = 360 - 22.3 = 337.7°$

Example 5: Solve each of the following trigonometric equations using exact values when possible on the interval $[0, 2\pi]$. radians

a. $5\sin\theta + 2 = 1 + 3\sin\theta$

$-3\sin\theta \qquad -3\sin\theta$

$2\sin\theta + 2 = 1$

$-2 \quad -2$

$\frac{2\sin\theta}{2} = \frac{-1}{2}$

$\sin\theta = -\frac{1}{2}$

$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

☆ b. $3\cos\theta - 1 = \cos\theta + 1$

$2\cos\theta = 2$

$\cos\theta = 1$

$\theta = 0, 2\pi$

Example 6: Solve each of the following trigonometric equations using exact values when possible on the interval $[0°, 360°]$.

a. $2\tan\theta + 2 = 2 + \tan\theta$

$-\tan\theta \qquad -\tan\theta$

$\tan\theta + 2 = 2$

$\tan\theta = 0$

$\theta = 0°, 180°, 360°$

b. $6\sin\theta + \sqrt{3} = 4\sin\theta$

$-4\sin\theta \qquad -4\sin\theta$

$2\sin\theta = -\sqrt{3}$

$\sin\theta = \frac{-\sqrt{3}}{2}$

$\theta = 240°, 300°$

Example 7: Solve the following trigonometric equation using exact values when possible on the interval $[0, 2\pi]$.

$$\tan^2\theta - 5\tan\theta + 4 = 0$$

$x^2 - 5x + 4 = 0$

$(x-1)(x-4) = 0$

$(\tan\theta - 1)(\tan\theta - 4) = 0$

$\tan\theta = 1 \qquad \tan\theta = 4$

$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

Use calc in Rad

$\theta = \tan^{-1}(4)$

$\theta_r \approx 1.33$

Q_1 & Q_3

$Q_1 \Rightarrow \theta = 1.33$

$Q_3 \Rightarrow \theta = \pi + 1.33$

$\theta \approx 4.47$

Example 8: Solve each the following trigonometric equation using exact values when possible on the interval $[0°, 360°]$.

$$\cos^2\theta + \cos\theta = 0$$

$x^2 + x = 0$

$x(x+1) = 0$

$\cos\theta(\cos\theta + 1) = 0$

$\cos\theta = 0 \qquad \cos\theta = -1$

$\theta = 90°, 270°, 180°$

Example 9: Solve each of the following trigonometric equations using exact values when possible on the interval $[0, 2\pi]$.

a. $\sec\theta = -2$

$\cos\theta = -\frac{1}{2}$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$\tan\frac{\pi}{6} \rightarrow \frac{y}{x} = \frac{1}{2} \div \frac{\sqrt{3}}{2}$

$= \frac{1}{2} \times \frac{2}{\sqrt{3}}$

$= \frac{1}{\sqrt{3}}$

$\therefore \tan\frac{\pi}{3} = \sqrt{3}$

b. $(\cot\theta - \sqrt{3})(\sqrt{2}\csc\theta + 2) = 0$

$\cot\theta = \sqrt{3} \qquad \sqrt{2}\csc\theta = -2$

$\tan\theta = \frac{1}{\sqrt{3}} \qquad \csc\theta = -\frac{2}{\sqrt{2}}$

$\theta = \frac{\pi}{6}, \frac{7\pi}{6} \qquad \sin\theta = -\frac{\sqrt{2}}{2}$

$\theta = \frac{5\pi}{4}, \frac{7\pi}{4}$

$\therefore \theta = \{\frac{\pi}{6}, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{7\pi}{4}\}$

Example 10: Solve the following trigonometric equation using exact values when possible on the interval [0°, 360°].

$$(\sec\theta - 4)(5\cot\theta - 3) = 0$$

$\sec\theta = 4$

$\cos\theta = \frac{1}{4}$

$\theta = \cos^{-1}\left(\frac{1}{4}\right)$

$\theta = 76°$

$\boxed{\theta = 76°, 284°}$

Cos is +'ve in 1 & 4

$5\cot\theta = 3$

$\cot\theta = \frac{3}{5}$

$\tan\theta = \frac{5}{3}$

$\theta = \tan^{-1}\left(\frac{5}{3}\right)$

$\theta_r = 59°$

$\boxed{\theta = 59°, 239°}$

Tan is +'ve in Q_1 & Q_3

$\theta = \{76°, 284°, 59°, 239°\}$

Example 11: Solve each of the following trigonometric equations using exact values when possible on the interval $\boxed{[0, 2\pi]}$.

Let $x = 2\theta$

∴ interval is doubled $[0, 4\pi]$

a. $2\sin 2\theta + 1 = 0$

$2\sin x + 1 = 0$

$2\sin x = -1$

$\sin x = -\frac{1}{2}$

$x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$

$\frac{7\pi}{6} + 2\pi = \frac{7\pi}{6} + \frac{12\pi}{6} = \frac{19\pi}{6}$

$\frac{11\pi}{6} + \frac{12\pi}{6} = \frac{23\pi}{6}$

$2\theta = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$

$\theta = \left\{\frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}\right\}$

☆ b. $\cos 3\theta = \frac{\sqrt{3}}{2}$

Let $3\theta = x$ + Domain is tripled $[0, 6\pi]$

$\cos x = \frac{\sqrt{3}}{2}$

$x = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}, \frac{25\pi}{6}, \frac{35\pi}{6}$

+ 2π + 2π + 2π + 2π

$3\theta = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}, \frac{25\pi}{6}, \frac{35\pi}{6}$

$\theta = \left\{\frac{\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{23\pi}{18}, \frac{25\pi}{18}, \frac{35\pi}{18}\right\}$

Example 12: Solve each of the following trigonometric equations using exact values when possible on the interval [0°, 360°].

a. $\sin 2\theta = 0.85$ Let $x = 2\theta$ $\therefore [0°, 720°]$

$\sin x = 0.85$

$x = \sin^{-1}(0.85)$

$x = 58°, 122°, 418°, 482°$

$\xrightarrow{+360°}$ $\xrightarrow{+360°}$

$2\theta = 58°, 122°, 418°, 482°$

$\theta = \{29°, 61°, 209°, 241°\}$

b. $\cos \frac{1}{2}\theta = \frac{\sqrt{3}}{2}$ Let $x = \frac{1}{2}\theta$ $\therefore$ Interval [0, 180]

$\cos x = \frac{\sqrt{3}}{2}$

$x = 30°, \cancel{330°}$ → Not in interval

$\therefore x = 30°$

$\frac{1}{2}\theta = 30°$

$\boxed{\theta = 60°}$

Example 13: Solve the following trigonometric equation using exact values when possible on the interval $(-\infty, \infty)$.

Give general solution

$2x^2 - x - 1 = 0$

$2\sin^2\theta - \sin\theta - 1 = 0$

$(2\sin\theta + 1)(\sin\theta - 1) = 0$

$2\sin\theta = -1$

$\sin\theta = -\frac{1}{2}$

$\theta = 330°, 210°$

$\sin\theta = 1$

$\theta = 90°$

$\{\theta = 90 + 360k, k \in I\}$

$\left\{ \begin{array}{l} \theta = 210° + 360k, k \in I \\ \theta = 330° + 360k, k \in I \end{array} \right\}$

$\theta = 90° + 360k, k \in I$

Example 14:

a. Use algebra to solve the equation $7 + 2\sin x = 4\sin x + 5$ for $\boxed{-360° < x \leq 0°}$, then write the general solution.

$7 + 2\sin x = 4\sin x + 5$

$-7 \quad -4\sin x \quad -4\sin x \quad -7$

$\frac{-2\sin x}{-2} = \frac{-2}{-2}$

$\sin x = 1$

$x = 90°, \; 90° - 360° = -270°$

$\therefore x = -270°$

General Solution: $x = -270° + 360°k, k \in I$

b. Use algebra to determine the general solution of the equation $\cos 3x = -1$ over the set of real numbers, then list the roots in the domain $-2\pi \leq x < 0$.

$\cos 3x = -1$ Let $\theta = 3x$ $\therefore$ Domain $[-6\pi, 0]$

$\cos\theta = -1$

$\theta = -\pi, -3\pi, -5\pi$

$x = -\frac{\pi}{3}, -\pi, -\frac{5\pi}{3}$

$\therefore \{x = -\frac{\pi}{3} + 2\pi k, k \in I\}$

Example 15: Use algebra to solve $2\cos^2 x = 1$ over the domain $0° \leq x < 360°$.

$2\cos^2 x = 1$

$\cos^2 x = \frac{1}{2}$

$\cos x = \sqrt{\frac{1}{2}}$

$\cos x = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{\sqrt{2}}{2}$

$\cos x = \frac{\sqrt{2}}{2}$

$\therefore x = \{45°, 315°\}$

$\boxed{\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}}$

$\frac{1}{2} = \frac{2}{4}$

$\sqrt{2}\sqrt{2} = \sqrt{2 \times 2}$

$= \sqrt{4}$

$= 2$

7.3 Reciprocal and Quotient Identities

A **trigonometric identity** is a statement that relates trigonometric ratios, and is true for all values of the variable for which the trigonometric ratios are defined.

Reciprocal Identities

$\csc\theta = \frac{1}{\sin\theta}, \sin\theta \neq 0$ $\qquad$ $\sec\theta = \frac{1}{\cos\theta}, \cos\theta \neq 0$

$\cot\theta = \frac{1}{\tan\theta}, \sin\theta \neq 0$

Quotient Identities

$\tan\theta = \frac{y}{x}$

$\tan\theta = \frac{\sin\theta}{\cos\theta}, \cos\theta \neq 0$ $\qquad$ $\cot\theta = \frac{\cos\theta}{\sin\theta}, \sin\theta \neq 0$

Strategies to Solve Identities

- keep the left hand side and the right hand side separate
- work on the side which is more expanded or complicated
- change $\sin\theta$ and $\cos\theta$ when necessary
- simplify by finding a common denominator

Example 1: For each identity below:

a. Verify the identity for $\theta = 30°$.
b. Prove the identity.

i) $(\sec\theta)(1+\cos\theta) = 1 + \sec\theta$

Verify:		Prove:
LHS	RHS	LHS:
$\sec 30°(1+\cos 30)$	$1+\sec\theta$	$\sec\theta(1+\cos\theta)$
$\frac{1}{\cos 30}(1+\cos 30)$	$1+\sec 30$	$\frac{1}{\cos\theta}(1+\cos\theta)$
$\frac{1}{\frac{\sqrt{3}}{2}}\left(1+\frac{\sqrt{3}}{2}\right)$	$1+\frac{1}{\cos 30}$	$\frac{1}{\cos\theta} + \frac{\cos\theta}{\cos\theta}$
$\frac{2}{\sqrt{3}}\left(1+\frac{\sqrt{3}}{2}\right)$	$1+\frac{1}{\frac{\sqrt{3}}{2}}$	$\sec\theta + 1$
$\frac{2}{\sqrt{3}} + \frac{\cancel{2\sqrt{3}}}{\cancel{\sqrt{3}(2)}}$	$1+\frac{2}{\sqrt{3}}$	$= \text{RHS}$
$\frac{2}{\sqrt{3}} + 1$		QED

LHS = RHS $\therefore$ QED

ii) $1 - \tan\theta = \frac{\cot\theta - 1}{\cot\theta}$

~~Verify~~: Prove:

$$RHS = \frac{\cot\theta - 1}{\cot\theta}$$

$$= \frac{\cot\theta}{\cot\theta} - \frac{1}{\cot\theta}$$

$$= 1 - \tan\theta$$

$$= LHS$$

QED

Example 2: For each identity below:

a. Determine the non-permissible values of θ.
b. Prove the identity.

i) $\frac{\cot\theta}{\csc\theta} = \cos\theta$

NPV: $\sin\theta \neq 0$

$\theta \neq 0, \pi, 2\pi \ldots$

$\theta \neq k\pi, k \in I$

$$LHS = \frac{\frac{\cos\theta}{\sin\theta}*}{\frac{1}{\sin\theta}*}$$

$$= \frac{\cos\theta}{\sin\theta} \div \frac{1}{\sin\theta}$$

$$= \frac{\cos\theta}{\cancel{\sin\theta}} \times \frac{\cancel{\sin\theta}}{1}$$

$$= \frac{\cos\theta}{1}$$

$$= \cos\theta$$

$$= RHS$$

QED

ii) $\cos\theta = \frac{1+\cos\theta}{1+\sec\theta}$

NPV: $1+\sec\theta \neq 0$

$\sec\theta \neq -1$

$\frac{1}{\cos\theta} \neq -1$

$\cos\theta \neq -1$

$\theta \neq \pi + 2\pi k,\ k \in I$

$$\begin{aligned}
RHS &= \frac{1+\cos\theta}{1+\sec\theta} \\
&= \frac{1+\cos\theta}{1+\frac{1}{\cos\theta}} \\
&= \frac{1+\cos\theta}{\frac{\cos\theta}{\cos\theta}+\frac{1}{\cos\theta}} \\
&= \frac{1+\cos\theta}{\frac{\cos\theta+1}{\cos\theta}} \\
&= 1+\cos\theta \div \frac{\cos\theta+1}{\cos\theta} \\
&= (\cancel{1+\cos\theta})\left(\frac{\cos\theta}{\cancel{\cos\theta+1}}\right) \\
&= \cos\theta \\
&= LHS
\end{aligned}$$

QED

Example 3: Prove the following identities.

a. $(1+\sin\theta)(1-\sin\theta) = \frac{1}{\sec^2\theta}$

$* (\sin\theta)^2 = \sin^2\theta$

LHS: $(1+\sin\theta)(1-\sin\theta)$

$= 1 - \sin\theta + \sin\theta - \sin^2\theta$

$= 1 - \sin^2\theta$

$= \cos^2\theta$

$= \frac{1}{\sec^2\theta}$

$= RHS$

$= QED$

$* \; x^2 + y^2 = 1$

$\cos^2 x + \sin^2 x = 1$

$\cos^2 x = 1 - \sin^2 x$

b. $\cos\theta\,(\tan\theta + \cot\theta) = \csc\theta$

LHS $= \cos\theta\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$

$= \cancel{\cos\theta}\left(\frac{\sin\theta}{\cancel{\cos\theta}}\right) + \cos\theta\left(\frac{\cos\theta}{\sin\theta}\right)$

$= \frac{\sin\theta^{\times\sin\theta}}{1_{\times\sin\theta}} + \frac{\cos^2\theta}{\sin\theta}$

$= \frac{\sin^2\theta}{\sin\theta} + \frac{\cos^2\theta}{\sin\theta}$

$= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta}$

$= \frac{1}{\sin\theta}$

$= \csc\theta = RHS$

QED

$x^2 + y^2 = 1$

$\sin^2 x + \cos^2 x = 1$

c. $\frac{\sin\theta}{\csc\theta} + \frac{\cos\theta}{\sec\theta} = 1$

*

$x^2 + y^2 = 1$

$\cos^2 x + \sin^2 x = 1$

LHS $= \dfrac{\sin\theta}{\frac{1}{\sin\theta}} + \dfrac{\cos\theta}{\frac{1}{\cos\theta}}$

$= \left(\sin\theta \div \frac{1}{\sin\theta}\right) + \left(\cos\theta \div \frac{1}{\cos\theta}\right)$

$= \sin\theta \times \sin\theta + \cos\theta \times \cos\theta$

$= \sin^2\theta + \cos^2\theta$

$= 1 =$ RHS

QED

d. $\csc\theta - \sin\theta = \cot\theta\cos\theta$

LHS:

$\csc\theta - \sin\theta$

$\frac{1}{\sin\theta} - \sin\theta\left(\frac{\sin\theta}{\sin\theta}\right)$

$\frac{1}{\sin\theta} - \frac{\sin^2\theta}{\sin\theta}$

$= \frac{1-\sin^2\theta}{\sin\theta}$

$= \frac{\cos^2\theta}{\sin\theta}$

RHS:

$\cot\theta\cos\theta$

$= \frac{\cos\theta}{\sin\theta}\left(\cos\theta\right)$

$= \frac{\cos^2\theta}{\sin\theta}$

$\therefore$ LHS = RHS

QED

* $\sin^2 x + \cos^2 x = 1$

$\cos^2 x = 1 - \sin^2 x$

Example 3: Use algebra to solve each equation over the domain $0 \le x < 2\pi$.

a. $2\sin x = 3 + 2\csc x$

$\left[2\sin x = 3 + 2\left(\frac{1}{\sin x}\right) \right] \sin x$

$2\sin^2 x = 3\sin x + 2$

$2\sin^2 x - 3\sin x - 2 = 0$

$(2\sin x + 1)(\sin x - 2) = 0$

$2\sin x + 1 = 0$ $\quad$ $\sin x - 2 = 0$

$\sin x = -\frac{1}{2}$ $\quad$ $\cancel{\sin x = 2}$

Cannot solve / not possible

$x = \left\{ \frac{7\pi}{6}, \frac{11\pi}{6} \right\}$

b. $\frac{\sin x}{\cos x} = \frac{\cos x}{\cos x}$

$\tan x = 1$

$x = \left\{ \frac{\pi}{4}, \frac{5\pi}{4} \right\}$

7.4 The Pythagorean Identities

Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$
$\tan^2\theta + 1 = \sec^2\theta$
$\cot^2\theta + 1 = \csc^2\theta$

Example 1: Prove each identity.

a. $\cot\theta + \tan\theta = \csc\theta\sec\theta$

$$LHS = \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\cos\theta(\cos\theta)}{\sin\theta(\cos\theta)} + \frac{\sin\theta(\sin\theta)}{\cos\theta\sin\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta\cos\theta}$$

$$= \left(\frac{1}{\sin\theta}\right)\left(\frac{1}{\cos\theta}\right)$$

$$= \csc\theta\sec\theta$$

$= RHS$

QED

b. $\cot^3\theta = \cot\theta\csc^2\theta - \cot\theta$

$* \cot^2\theta + 1 = \csc^2\theta$

$\cot^2\theta = \csc^2\theta - 1$

$$\begin{aligned}
RHS &= \cot\theta(\csc^2\theta - 1) \\
&= \cot\theta(\cot^2\theta) \\
&= \cot^3\theta \\
&= LHS
\end{aligned}$$

QED

c. $\csc\theta\cos^2\theta + \sin\theta = \csc\theta$

$$\begin{aligned}
LHS:\ & \frac{1}{\sin\theta}(\cos^2\theta) + \sin\theta \\
&= \frac{\cos^2\theta}{\sin\theta} + \sin\theta\left(\frac{\sin\theta}{\sin\theta}\right) \\
&= \frac{\cos^2\theta}{\sin\theta} + \frac{\sin^2\theta}{\sin\theta} \\
&= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta} \\
&= \frac{1}{\sin\theta} \\
&= \csc\theta \\
&= RHS
\end{aligned}$$

QED

d. $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta$

$x^4 - y^4$

$$LHS = (\sin^2\theta + \cos^2\theta)(\sin^2\theta - \cos^2\theta)$$
$$= \sin^2\theta - \cos^2\theta$$
$$= RHS$$

QED

Example 2: Prove each identity.

$\sin^2 x + \cos^2 x = 1$
$\sin^2 x = 1 - \cos^2 x$

a. $\frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$

$$RHS = \frac{\sin\theta}{1+\cos\theta}\left(\frac{1-\cos\theta}{1-\cos\theta}\right)$$
$$= \frac{\sin\theta(1-\cos\theta)}{1-\cos\theta+\cos\theta-\cos^2\theta}$$
$$= \frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta}$$
$$= \frac{\cancel{\sin\theta}(1-\cos\theta)}{\sin^{\cancel{2}}\theta}$$
$$= \frac{1-\cos\theta}{\sin\theta}$$
$$= LHS \qquad QED$$

b. $\frac{1}{1-\cos\theta}+\frac{1}{1+\cos\theta}=2\csc^2\theta$ LCD = $(1-\cos\theta)(1+\cos\theta)$

$$LHS = \frac{1(1+\cos\theta)}{(1-\cos\theta)(1+\cos\theta)} + \frac{1(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}$$

$$= \frac{1+\cos\theta+1-\cos\theta}{1-\cos^2\theta}$$

$$= \frac{2}{1-\cos^2\theta}$$

$$= \frac{2}{\sin^2\theta}$$

$$= 2\left(\frac{1}{\sin^2\theta}\right)$$

$$= 2\csc^2\theta$$

$= RHS$

QED

c. $\frac{\cos\theta\sin\theta}{1+\sin\theta}=\frac{1-\sin\theta}{\cot\theta}$

$$LHS = \frac{\cos\theta\sin\theta}{1+\sin\theta}$$

$$= \frac{\cos\theta\sin\theta}{1+\sin\theta}\left(\frac{1-\sin\theta}{1-\sin\theta}\right)$$

$$= \frac{\cos\theta\sin\theta(1-\sin\theta)}{1-\sin^2\theta}$$

$$= \frac{\cancel{\cos\theta}\sin\theta(1-\sin\theta)}{\cos^{\cancel{2}}\theta}$$

$$= \frac{\sin\theta(1-\sin\theta)}{\cos\theta}$$

$$= \tan\theta(1-\sin\theta)$$

$$= \frac{1}{\cot\theta}(1-\sin\theta)$$

$$= \frac{1-\sin\theta}{\cot\theta} = RHS$$

d. $\frac{\cos\theta}{1-\sin\theta}+\frac{\cos\theta}{1+\sin\theta}=2\sec\theta$

$$LHS = \frac{\cos\theta}{1-\sin\theta}+\frac{\cos\theta}{1+\sin\theta}$$

$$= \frac{\cos\theta(1+\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}+\frac{\cos\theta(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}$$

$$= \frac{\cos\theta+\cancel{\sin\theta\cos\theta}+\cos\theta-\cancel{\sin\theta\cos\theta}}{(1-\sin\theta)(1+\sin\theta)}$$

$$= \frac{2\cos\theta}{1-\sin^2\theta}$$

$$= \frac{2\cancel{\cos\theta}}{\cos^{\cancel{2}}\theta} = \frac{2}{\cos\theta} = 2\sec\theta = RHS$$

QED

Example 3: Solve over the $[0, 2\pi]$.

$\sin^2 x = 1-\cos^2 x$

a. $3-3\cos x-2\sin^2 x=0$

$$3-3\cos x-2(1-\cos^2 x)=0$$

$$3-3\cos x-2+2\cos^2 x=0$$

$$2\cos^2 x-3\cos x+1=0$$

$$(2\cos x-1)(\cos x-1)=0$$

$$2\cos x=1 \qquad \cos x=1$$

$$\cos x=\frac{1}{2}$$

$$x=\left\{\frac{\pi}{3}, \frac{5\pi}{3}, 0, 2\pi\right\}$$

b. $2\cos^2 x - 3\sin x = 0$ $\qquad$ $\cos^2 x = 1 - \sin^2 x$

$2(1 - \sin^2 x) - 3\sin x = 0$

$2 - 2\sin^2 x - 3\sin x = 0$

$-2\sin^2 x - 3\sin x + 2 = 0$

$2\sin^2 x + 3\sin x - 2 = 0$

$(2\sin x - 1)(\sin x + 2) = 0$

$2\sin x = 1$ $\qquad$ $\sin \cancel{x =} -2$

$\sin x = \frac{1}{2}$ $\qquad$ Not possible.

$x = \{\frac{\pi}{6}, \frac{5\pi}{6}\}$

c. $2\tan^2\theta + \sec\theta + 1 = 0$ $\qquad$ $\tan^2\theta + 1 = \sec^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

$2(\sec^2\theta - 1) + \sec\theta + 1 = 0$

$2\sec^2\theta - 2 + \sec\theta + 1 = 0$

$2\sec^2\theta + \sec\theta - 1 = 0$

$(2\sec\theta - 1)(\sec\theta + 1) = 0$

$\sec\theta = \frac{1}{2}$ $\qquad$ $\sec\theta = -1$ $\qquad$ $\frac{1}{-1} = -1$

$\cos \cancel{\theta =} 2$ $\qquad$ $\cos\theta = -1$

Not possible. $\qquad$ $\theta = \{\pi\}$

7.5 Sum and Difference Identities

Sum and Difference Identities

α = alpha
β = beta

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
$\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

Example 1: Write the expression in simplest form, then evaluate where possible.

$\alpha = 8x \quad \beta = 3x$

a. $\sin 8x\cos 3x - \cos 8x\sin 3x$

$\sin\alpha\cos\beta - \cos\alpha\sin\beta = \sin(\alpha-\beta)$
$= \sin(8x-3x)$
$= \sin(5x)$

b. $\frac{\left(\tan\frac{\pi}{6}+\tan\frac{\pi}{12}\right)}{1-\tan\frac{\pi}{6}\tan\frac{\pi}{12}}$ $\quad \alpha = \frac{\pi}{6}$ $\quad \beta = \frac{\pi}{12}$

$= \tan(\alpha+\beta)$
$= \tan\left(\frac{\pi}{6}+\frac{\pi}{12}\right)$
$= \tan\left(\frac{2\pi}{12}+\frac{\pi}{12}\right)$
$= \tan\left(\frac{3\pi}{12}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

c. $\cos 33^\circ\cos 27^\circ - \sin 33^\circ\sin 27^\circ = \cos(33+27)$
$= \cos(60^\circ)$
$= \frac{1}{2}$

Example 2: Determine the exact value.

* Break up the given angle in terms of angles you know a/a on unit circle.

a. $\sin 105°$

$= \sin(45° + 60°)$

$\alpha = 45°$
$\beta = 60°$

$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \sin 45° \cos 60° + \cos 45° \sin 60°$

$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{2}+\sqrt{6}}{4}$

b. $\cos\frac{11\pi}{12}$ $= \cos\left(\frac{9\pi}{12} + \frac{2\pi}{12}\right) = \cos\left(\frac{3\pi}{4} + \frac{\pi}{6}\right)$

$= \cos\left(\frac{3\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{3\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$

$= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$

$= \frac{-\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{-\sqrt{6}-\sqrt{2}}{4}$

Example 3: Prove this identity: $\sin(\pi - x) = \sin x$

$\alpha = \pi$
$\beta = x$

Use sin difference formula.

$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= \sin\pi\cos x - \cos\pi\sin x$

$= (0)\cancel{\cos x} - (-1)\sin x$

$= \sin x = \text{RHS}$

$\frac{8\pi}{12} = \frac{2\pi}{3}$

$\frac{9\pi}{12} = \frac{3\pi}{4}$

$\frac{10\pi}{12} = \frac{5\pi}{6}$

$\frac{12\pi}{12} = \pi$

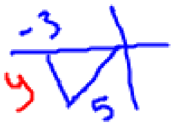
$\frac{\pi}{2} = \frac{6\pi}{12}$

$\frac{\pi}{3} = \frac{4\pi}{12}$

$\frac{\pi}{4} = \frac{3\pi}{12}$

$\frac{\pi}{6} = \frac{2\pi}{12}$

Example 4: Given angle α in standard position with its terminal arm in Quadrant 3 and $\cos\alpha = -\frac{3}{5}$, and angle β in standard position with its terminal arm in Quadrant 2 and $\sin\beta = \frac{1}{3}$, determine the exact value of $\sin(\alpha+\beta)$.



$\cos\alpha = -\frac{3}{5} = \frac{x}{r}$

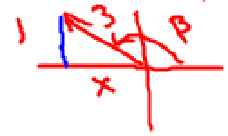
$\therefore r = 5$

$x = -3$

$y = \sqrt{5^2-3^2} = \sqrt{25-9} = \sqrt{16} = -4$

$\sin\alpha = \frac{y}{r} = -\frac{4}{5}$

$\sin\beta = \frac{1}{3} = \frac{y}{r}$ $\quad y = 1$, $r = 3$



$x = \sqrt{3^2-1^2}$

$x = \sqrt{9-1}$

$x = -\sqrt{8}$

$\therefore \cos\beta = \frac{x}{r} = \frac{-\sqrt{8}}{3}$

$\therefore \sin(\alpha+\beta)$

$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \left(-\frac{4}{5}\right)\left(-\frac{\sqrt{8}}{3}\right) + \left(-\frac{3}{5}\right)\left(\frac{1}{3}\right)$

$= \dfrac{4\sqrt{8}-3}{15}$

Example 5: Solve the equation $\underbrace{\cos 4x\cos x + \sin 4x \sin x}_{\cos(\alpha-\beta)} = 1$ over the domain $0 \le x < 2\pi$.

$\cos(4x - x) = 1$

$\cos 3x = 1$ $\qquad$ Let $\theta = 3x$

$\cos\theta = 1$ $\qquad$ But Domain is tripled $[0, 6\pi]$

$\theta = 0, 2\pi, 4\pi, 6\pi$

$\frac{3x}{3} = \frac{0, 2\pi, 4\pi, 6\pi}{3}$

$x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{6\pi}{3}$

$x = \{0, \frac{2\pi}{3}, \frac{4\pi}{3}\}$ $\cancel{2\pi}$

#1, 2, 4, 5b, 6, 7, 12a

7.6 Double-Angle Identities

Double Angle Identities

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$$\cos 2\alpha = \begin{cases} \cos^2\alpha - \sin^2\alpha \\ 2\cos^2\alpha - 1 \\ 1 - 2\sin^2\alpha \end{cases}$$

$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$

Example 1: Write each expression as a single trigonometric ratio, then evaluate where possible.

a. $\cos^2\left(\frac{\pi}{4}\right) - \sin^2\left(\frac{\pi}{4}\right)$ $= \cos(2\alpha)$ when $\alpha = \frac{\pi}{4}$

$= \cos\left(2\left(\frac{\pi}{4}\right)\right)$

$= \cos\left(\frac{\pi}{2}\right) = 0$

b. $\frac{2\tan\left(\frac{\pi}{6}\right)}{\tan^2\left(\frac{\pi}{6}\right)-1}$ $= \frac{2\tan\left(\frac{\pi}{6}\right)}{-1+\tan^2\left(\frac{\pi}{6}\right)} = \frac{2\tan\left(\frac{\pi}{6}\right)}{-1\left(1-\tan^2\left(\frac{\pi}{6}\right)\right)} = -\left(\frac{2\tan\left(\frac{\pi}{6}\right)}{1-\tan^2\left(\frac{\pi}{6}\right)}\right)$

$= -\tan\left(2\cdot\frac{\pi}{6}\right)$

$= -\tan\left(\frac{\pi}{3}\right)$

$= -\sqrt{3}$

c. $2\sin 5t\cos 5t$ $= \sin 2\alpha$ where $\alpha = 5t$

$= \sin(2\cdot 5t)$

$= \sin(10t)$

Example 2: Given angle θ is in standard position with its terminal arm in Quadrant 4 and $\cos\theta = \frac{2}{5}$, determine the exact value of each trigonometric ratio.

$\sin\theta = \frac{-\sqrt{21}}{5}$ $\quad \cos\theta = \frac{2}{5} = \frac{x}{r}$

a. $\sin 2\theta$
b. $\cos 2\theta$

$y = \sqrt{5^2 - 2^2}$

$y = \sqrt{25-4}$

$y = -\sqrt{21}$

a) $\sin 2\theta = 2\sin\theta\cos\theta$

$= 2\left(-\frac{\sqrt{21}}{5}\right)\left(\frac{2}{5}\right)$

$= -\frac{4\sqrt{21}}{25}$

b) $\cos 2\theta = 2\cos^2\theta - 1$

$= 2\left(\frac{2}{5}\right)^2 - 1$

$= 2\left(\frac{4}{25}\right) - 1$

$= \frac{8}{25} - 1$

$= \frac{8}{25} - \frac{25}{25}$

$= -\frac{17}{25}$

Example 3: Prove each identity.

a. $\cot\theta = \frac{\cos 2\theta + 1}{\sin 2\theta}$

* We want $\cos\theta$ in numerator $\therefore$ pick $\cos 2\theta = 2\cos^2\theta - 1$

$$RHS = \frac{\cos 2\theta + 1}{\sin 2\theta}$$

$$= \frac{(2\cos^2\theta - 1) + 1}{2\sin\theta\cos\theta}$$

$$= \frac{\cancel{2}\cos^{\cancel{2}}\theta}{\cancel{2}\sin\theta\cancel{\cos\theta}}$$

$$= \frac{\cos\theta}{\sin\theta}$$

$$= \cot\theta$$

$$= LHS$$

b. $\cot\theta\csc 2\theta = \frac{1}{2\sin^2\theta}$

* Start on side w/ double angle

$$LHS = \cot\theta\csc 2\theta$$

$$= \left(\frac{\cos\theta}{\sin\theta}\right)\left(\frac{1}{\sin 2\theta}\right)$$

$$= \left(\frac{\cancel{\cos\theta}}{\sin\theta}\right)\left(\frac{1}{2\sin\theta\cancel{\cos\theta}}\right)$$

$$= \frac{1}{2\sin\theta\sin\theta}$$

$$= \frac{1}{2\sin^2\theta} = RHS$$

Example 3: Solve the equation $\frac{1}{2}\sin 2x - \cos^2 x = 0$ over the domain $0 \le x < 2\pi$.

Change $\sin 2x$ to $2\sin x\cos x$

$$\cancel{\tfrac{1}{2}}(\cancel{2}\sin x\cos x) - \cos^2 x = 0$$

$$\sin x\cos x - \cos^2 x = 0$$

$$(\cos x)(\sin x - \cos x) = 0$$

$\therefore \cos x = 0$ or $\sin x - \cos x = 0$

$x(x-2)=0$
$x=0$
$x-2=0$

$x = \{\frac{\pi}{2}, \frac{3\pi}{2}\}$

$$\sin x - \cos x = 0$$

$$\frac{\sin x}{\cos x} = \frac{\cos x}{\cos x}$$

$$\tan x = 1$$

$$x = \{\frac{\pi}{4}, \frac{5\pi}{4}\}$$

$$\therefore x = \{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{4}, \frac{5\pi}{4}\}$$