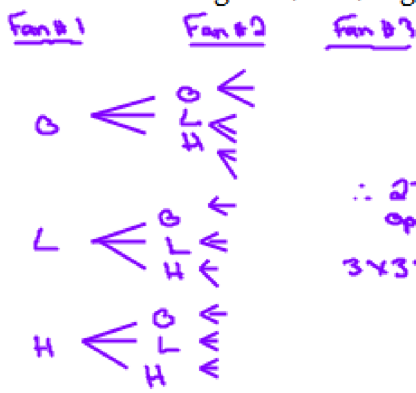
Chapter 8 Permutations and Combinations

8.1 The Fundamental Counting Principle

O L H

Example 1: A fan has 3 settings: off, low, high. How many ways are there to set 3 fans?



∴ 27 options

$3 \times 3 \times 3 = 27$

Example 2: At a grocery store there are 2 line ups, one has 2 people in line and the other has 3 people in line. If they were to rearrange themselves but stayed in the same line, how many different possible line ups are there?

Line 1 - 2 people (A+B) → AB, BA 2

Line 2 - 3 people (x y z) xyz, xzy, yxz, yzx, zyx, zxy 6

$\therefore 2 \times 6 = 12$ options

The Fundamental Counting Principle

If there are n_1 different objects in one set and n_2 different objects in a second set, then the number of ways of choosing one object from each set is $n_1 n_2$.

In other words if one event can occur in a ways and a second event can occur in b ways then the number of ways that both events can occur is ab ways.

Example 3: For an online banking account, the minimum security standards require a password to have 2 letters followed by 5 digits. All letters and digits may be used more than once. How many passwords are possible?

Letters = 26
Digits = 10

$\underset{\text{letter}}{\underline{26}} \times \underset{\text{Letter}}{\underline{26}} \times \underset{\text{Dig.}}{\underline{10}} \times \underset{\text{D.}}{\underline{10}} \times \underset{\text{D.}}{\underline{10}} \times \underset{\text{D.}}{\underline{10}} \times \underset{\text{D.}}{\underline{10}}$

$= 67,600,000$

Example 4: How many different license plates are possible if 3 letters must be followed by 3 numbers?

$\underset{L}{\underline{26}} \times \underset{L}{\underline{26}} \times \underset{L}{\underline{26}} \times \underset{N}{\underline{10}} \times \underset{N}{\underline{10}} \times \underset{N}{\underline{10}}$

$= 17,576,000$

Example 5: How many 4 digit even numbers can be made using the digits 2, 3, 4, 5, 6, 7, 8?
(if repetition is not allowed)

2, 3, 4, 5, 6, 7, 8

*even #
∴ has to end w/ 2, 4, 6, 8

$\underset{\text{6 options left}}{\underline{6}} \times \underline{5} \times \underline{4} \times \underset{\text{even}}{\underline{4}}$

* you can not repeat digits

∴ 480 options

Example 6: How many 5 letter "words" can be formed using the letters in the word FACTOR?
arrangement of letters

FACTOR = 6 letters * cannot repeat letters

$\underline{6} \times \underline{5} \times \underline{4} \times \underline{3} \times \underline{2} = \boxed{720}$

Example 7: How many 6 letter words beginning with F and ending in a vowel can be formed using the letters in the word FACTOR?
*no repetition

$\underset{F}{\underline{1}} \times \underline{4} \times \underline{3} \times \underline{2} \times \underline{1} \times \underset{\text{Vowel}}{\underline{2}} = \boxed{48}$

402 , ~~042~~

Example 8: How many 5 digit even numbers can be made using the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, 0?
*No zeros at the front of a # *

1, 2, 3, 4, 5, 6, 7, 8, 9, 0 *no rep*

Case 1: (zero is at the end) $\underline{9} \times \underline{8} \times \underline{7} \times \underline{6} \times \underset{\phi \text{ (zero)}}{\underline{1}} = \boxed{3024}$

Case 2: (2, 4, 6, 8 are at the end) $\underset{\text{can't be zero!}}{\underline{8}} \times \underline{8} \times \underline{7} \times \underline{6} \times \underset{2,4,6,8}{\underline{4}} = \boxed{10752}$

$\therefore$ 3024 + 10752
= 13776 possibilities

8.2 Permutations of Different Objects

Permutation: an arrangement of a set of objects where the order of the objects affects the total number of arrangements.

Example 1: How many ways can the letters in the word MATH be arranged?

Example 2: How many ways can the letters in the word COUNTED be arranged?

Factorial notation can be used to simplify these calculations.

The factorial sign mean to take the product of all natural numbers less than or equal to the given number.

$$n! = n(n-1)(n-2)(n-3)\ldots(3)(2)(1)$$

Factorial = !

$4! = 4 \times 3 \times 2 \times 1$

By definition $\boxed{0! = 1}$

Example 3: Determine the value of the following.

$7! =$ 5040

$3! =$ 6

$12! =$ 479001600

$(-2)! =$ undefined

$0! =$ 1

$1! =$ 1

* $\frac{8!}{6!} = \frac{8\times7\times\cancel{6}\times\cancel{5}\times\cancel{4}\times\cancel{3}\times\cancel{2}\times\cancel{1}}{\cancel{6}\times\cancel{5}\times\cancel{4}\times\cancel{3}\times\cancel{2}\times\cancel{1}}$

$= 8\times7 = 56$

* $\frac{(n+1)!}{(n-1)!} = \frac{(n+1)(n)\cancel{(n-1)!}}{\cancel{(n-1)!}}$

$= (n+1)(n)$

$= n^2+n$

Example 4: A puzzle designer decides to scramble the letters in the word EDUCATION to create a jumble puzzle. How many 9-letter permutations of EDUCATION can be created?

Example 5: How many 6 letter permutations are there using the letters ABCDEF?

8.2 Permutations of Different Objects

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Example 1: How many ways can the letters in the word MATH be arranged?

$\underline{4} \times \underline{3} \times \underline{2} \times \underline{1} = 4!$

$= 24$

Example 2: How many ways can the letters in the word COUNTED be arranged?

$7! = 5040$

Factorial notation can be used to simplify these calculations.

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Example 3: Determine the value of the following.

$7! =$

$3! =$

$12! =$

$(-2)! =$

$0! =$

$1! =$

$\frac{8!}{6!} =$

$\frac{(n+1)!}{(n-1)!} =$

Example 4: A puzzle designer decides to scramble the letters in the word EDUCATION to create a jumble puzzle. How many 9-letter permutations of EDUCATION can be created?

$\underline{9}\times\underline{8}\times\underline{7}\times\underline{6}\times\underline{5}\times\underline{4}\times\underline{3}\times\underline{2}\times\underline{1} = 9! = 362\,880$

Example 5: How many 6 letter permutations are there using the letters ABCDEF?

$6! = 720$

If all of the objects are not chosen to be arranged then the following formula can be used.

Permutation of Different Objects

The number of permutations of n distinct objects taken r at a time is:

You are ordering r, * out of n objects
total

$$_nP_r = \frac{n!}{(n-r)!}, n \geq r$$

Example 6: Determine the value of the following.

$_5P_3 =$ You are ordering 3 objects out of a collection of 5.

$$_5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5\times4\times3\times\cancel{2}\times\cancel{1}}{\cancel{2}\times\cancel{1}} = 5\times4\times3 = 60$$

$_8P_4 =$

$$_8P_4 = \frac{8!}{(8-4)!} = \frac{8!}{4!} = 1680$$

$_7P_1 =$

$$_7P_1 = \frac{7!}{(7-1)!} = \frac{7!}{6!} = 7$$

$_6P_0 =$ 1

Example 7: Eight students are competing in a 200m race. How many ways can the students finish first, second, and third?

Old Way:

$\underline{8} \times \underline{7} \times \underline{6} = 336$

$\boxed{\text{New Way:}}$

$$_nP_r = \frac{n!}{(n-r)!}$$

$$_8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = 336$$

Example 8: Of the 12 songs on your all-time favorite list, how many ways can you listen to 4 of them?

$_{12}P_4 = 11880$

Example 9: Solve each equation for n or r.

a. $_nP_2 = 56$

$\frac{n!}{(n-2)!} = 56$

$\frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} = 56$

$n(n-1) = 56$

$n^2 - n - 56 = 0$

$(n-8)(n+7) = 0$

$\boxed{n = 8}, \cancel{-7}$

b. $_5P_r = 20$

$_5P_1 = 5$

$\boxed{_5P_2 = 20}$ $\therefore r = 2$

$_5P_3 =$

$_5P_4 =$

$_5P_5 =$

8.3 Permutations Involving Identical Objects

Example 1: How many ways can the letters in word ELLA be arranged?

EL_1L_2A } These are
EL_2L_1A } the same.

2 L's

$$\frac{4!}{\text{the \# of ways to arrange Ls}} = \frac{4!}{2!} = 12$$

Permutations of Identical Objects

The number of permutations of n objects with r identical objects is: $\frac{n!}{r!}$.

Example 2: There are 7 boxes of cereal on a shelf. Five of the boxes are bran cereal, one box is puffed wheat, and the other box is granola. How many ways can the boxes be arranged in a row?

$$\frac{7!}{5!} = 7 \times 6 = 42$$

the # of ways to arrange 5 boxes of bran cereal

Example 3: Graeme walks 8 blocks from his home to the library. He always walks 4 blocks east and 4 blocks south. How many ways can Graeme walk to the library?

EEEE SSSS

$$\frac{8!}{4!\,4!} = 70$$

East South

Example 4: A kabob recipe requires 2 mushrooms, 2 shrimp, 2 cherry tomatoes, and 2 zucchini slices. How many ways can Amelie arrange these items on a skewer?

$$\frac{8!}{\underset{\text{mushrooms}}{2!}\,\underset{\text{shrimp}}{2!}\,\underset{\text{tomatos}}{2!}\,\underset{\text{zucchini}}{2!}} = 2520$$

Example 5: How many arrangements are there using the letters in the word MISSISSIPPI?

$$\frac{11!}{4!\,4!\,2!} = 34650$$

I → 4
S → 4
P → 2

Example 6: How many arrangements are there using the digits: 1, 1, 2, 2, 2, 3, 4, 4, 5, 5, 5, 5, 5?

$$\frac{13!}{2!\,3!\,2!\,5!} = 2162160$$

$$\boxed{\begin{array}{l} 1s \to 2 \\ 2s \to 3 \\ 4s \to 2 \\ 5s \to 5 \end{array}}$$

* Seating Arrangement Problem*

You and 5 friends are going to the movies. However, your ex is going & you can't sit beside them. How many ways are there to sit in a row of 6?

(Total Arrangements) - (You and your ex sitting side by side)

6!

_ _ _ _ _ _

$\underset{\text{you}}{\underline{2}}\ \underset{\text{ex}}{\underline{1}}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1}$

There are 5 locations for you & your ex to sit

$\therefore 5(2\times1\times4\times3\times2\times1)$

$5(2!)(4!)$

$= 240$

$6! - 240 = 720 - 240 = \boxed{480}$

8.4 Combinations

Example 1: In how many ways can 3 people be chosen to sit on a committee from a group of ~~5~~ 4 people? List the combinations.

ABCD

- ABC
- BCD
- ACD = CAD
- ABD

$4C3 = 4$

$\therefore$ 4 ways

When the arrangement of objects is important the arrangement is called a **permutation**.

When the arrangement of objects is not important the arrangement is called a **combination.**

Combination of Different Objects

The number of combinations of n distinct objects taken r at a time is:

$$*\quad {}_nC_r = \frac{n!}{r!(n-r)!},\ n \geq r$$

$$nPr = \frac{n!}{(n-r)!}$$

${}_nC_r$ can also be written using the notation $\binom{n}{r}$. Read as 'n choose r'.

$$nCr = \binom{n}{r}$$

Example 2: Type in the following.

$_nC_r = \frac{n!}{r!(n-r)!}$

$_5C_3 = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} = 10$

$_8C_4 = \frac{8!}{4!(8-4)!} = 70$

$_7C_1 = 7$

$_6C_0 = 1$

Example 3: In the Keno lottery, 20 numbers from 1 to 80 are chosen. How many combinations of 20 numbers are possible?

$_{80}C_{20} = 3.5 \times 10^{18}$

Example 4: How many different tickets are possible when playing the game Lotto 649?

$_{49}C_6 = 13,983,816$

Example 5: A local arena has 10 applicants interested in working in the snack bar.

a) How many ways can 4 applicants be chosen?

$_{10}C_4 = 210$

$$\frac{10!}{4!(10-4)!} = \frac{10!}{4!6!}$$

b) How many ways can 6 applicants be chosen?

$_{10}C_6 = 210$

$$\frac{10!}{6!(10-6)!} = \frac{10!}{6!4!}$$

Example 6: A new store must hire 3 cashiers and 4 stock clerks. There are 7 applicants for cashier and 8 applicants for store clerk. How many ways can the 7 employees be chosen?

$$(\underbrace{_7C_3}_{\text{cashier}}) \times (\underbrace{_8C_4}_{\text{clerk}})$$

$35 \times 70 = 2450$

Example 7 (An "At-Least" Problem)

There are 5 girls and 5 boys trying out for the water polo team. The team needs at least 3 girls at a time. How many combinations are there if there are 7 players on a team?

3 girls $(_5C_3)(_5C_4) = 10 \times 5 = 50$

or 4 girls $(_5C_4)(_5C_3) = 5 \times 10 = 50$

or 5 girls $(_5C_5)(_5C_2) = 1 \times 10 = 10$

$\boxed{110}$

8.5 Pascal's Triangle

Consider this arrangement of the numbers. It is called Pascal's Triangle.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Note the following patterns.

- Each number is the sum of the 2 numbers above it in the previous row. The outside numbers are always 1.
- All entries can be written using combinations.

row 1 = $\binom{0}{0} = {}_0C_0$

row2 = ${}_1C_0 = \binom{1}{0}$ $\binom{1}{1} = {}_1C_1$

row 3 = $\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2} = {}_2C_2$

$\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

$\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$

1

1 1

1 2 1

1 3 3 1

Note the following patterns.

In $\binom{n}{r}$, n is 1 less than the row number.

In $\binom{n}{r}$, r is 1 less than its position in the row.

8.6 The Binomial Theorem

Consider the following expansions.

$$\begin{aligned}
(a+b)^0 &= 1 && = 1 \text{ term} \\
(a+b)^1 &= 1a + 1b && = 2 \text{ terms} \\
(a+b)^2 &= 1a^2 + 2ab + 1b^2 && = 3 \text{ terms} \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
(a+b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5
\end{aligned}$$

Compare the expansions to Pascal's Triangle as numbers and combinations.

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1
\end{array}
\qquad
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}
\end{array}$$

Note the following patterns.

1) The coefficients of the expansion are the numbers in Pascal's Triangle.

2) When the exponent is n there are $n+1$ terms.

3) The first term is a^n and then the exponent of the power decreases by 1 for each subsequent term down to a^0.

4) The first term is b^0 and then the exponent of the power increases by 1 for each subsequent term up to b^n.

5) The sum of the exponents in each term is n.

The Binomial Theorem Using Combinations

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n$$

The Binomial Theorem Using Algebraic Expressions

$$(x+y)^n = x^n y^0 + n x^{n-1} y^1 + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots + x^0 y^n$$

Example 1: Expand and the simplify the first 4 terms of $(x+y)^8$.

$\binom{8}{0}(x)^8(y)^0 + \binom{8}{1}(x)^7(y)^1 + \binom{8}{2}x^6y^2 + \binom{8}{3}x^5y^3$

$8C_0\, x^8 y^0 + 8C_1\, x^7 y^1 + 8C_2\, x^6 y^2 + 8C_3\, x^5 y^3$

$1x^8 + 8x^7y + 28x^6y^2 + 56x^5y^3$

Example 2: Expand and then simplify $(\underset{x}{3b} - \underset{y}{1})^4$. $\Rightarrow$ 5 terms

$\binom{4}{0}(3b)^4(-1)^0 + \binom{4}{1}(3b)^3(-1)^1 + \binom{4}{2}(3b)^2(-1)^2 + \binom{4}{3}(3b)^1(-1)^3 + \binom{4}{4}(3b)^0(-1)^4$

$= 1(81)b^4 + 4(27)b^3(-1) + 6(9)b^2 + 4(3)b(-1) + 1(1)(1)$

$= 81b^4 - 108b^3 + 54b^2 - 12b + 1$

Example 3: Expand and then simplify $(2c^4 - 1)^3$.

$\binom{3}{0}(2c^4)^3(-1)^0 + \binom{3}{1}(2c^4)^2(-1)^1 + \binom{3}{2}(2c^4)^1(-1)^2 + \binom{3}{3}(2c^4)^0(-1)^3$

$1(8)c^{12} + 3(4)c^8(-1) + 3(2c^4) + 1(1)(-1)$

$8c^{12} - 12c^8 + 6c^4 - 1$

The exponent on the second variable of any term is called the **key number** and is always 1 less than the term number.

General Term in the Expansion of $(x + y)^n$

The general term, or $\overset{k+1}{k^{th}}$ term, in the expansion of $\overset{(a+b)^n}{(x + y)^n}$ is:

$\cancel{{}_nC_{k-1}x^{n-(k-1)}y^{k-1}}$. $t = \text{term}$

$t_{k+1} = {}_nC_k\, a^{n-k}\, b^k$

Example 4: Determine the 9th term in the expansion of $(x - 2)^{10}$

$t_9 = t_{8+1}$

$\therefore k = 8$

$n = 10$

$a = x$

$b = -2$

$t_9 = {}_{10}C_8\,(x)^{10-8}(-2)^8$

$= 45x^2(256)$

$= \boxed{1152\text{0}\,x^2}$

Example 5: Determine the 6th term in the expansion of $(m + 2y)^{15}$.

$t_{k+1} = {}_nC_k\, a^{n-k} b^k$

$t_6 = t_{5+1}$ $\quad \therefore k = 5 \quad n = 15 \quad a = m \quad b = 2y$

$t_6 = {}_{15}C_5\,(m)^{15-5}(2y)^5$

$= 3003m^{10}(32)y^5$

$= 96096m^{10}y^5$

Example 6: Determine the 5^{th} term in the expansion of $(3x + y^2)^{17}$.

$t_{k+1} = {}_nC_k (a)^{n-k} (b)^k$

$k = 4 \quad n = 17 \quad a = 3x \quad b = y^2$

$t_5 = {}_{17}C_4 (3x)^{17-4} (y^2)^4$

$= 2380(1594323)\, x^{13} y^8$

$= 3794488740\, x^{13} y^8$

Example 7: Find the term in the expansion of $(x + y)^7$ that contains x^2.

$t_{k+1} = {}_nC_k\, a^{n-k} b^k$

$k = ?\ 5 \quad n = 7 \quad a = x \quad b = y$

$= {}_7C_k\, x^{7-k} y^k$ We want $x^2 \therefore k = 5$

$t_{5+1} = {}_7C_5\, x^2 y^5$

$t_6 = 21x^2y^5$

Example 8: Find the term in the expansion of $(x + y)^{10}$ that contains y^8.

$t_{k+1} = {}_nC_k a^{n-k} b^k$

$k = ?\ 8 \quad n = 10 \quad a = x \quad b = y$

$t_{k+1} = {}_{10}C_k (x)^{10-k} y^k$ We want $y^8 \therefore k = 8$

$t_9 = {}_{10}C_8\, x^{10-8} y^8$

$t_9 = 45x^2y^8$

Example 9: Find the term in the expansion of $(x + y)^{10}$ that contains x^3y^7.

$k = 7 \quad n = 10 \quad a = x \quad b = y$

$t_{k+1} = {}_nC_k\, a^{n-k} b^k$

$= {}_{10}C_7\, x^{10-7} y^7$

$\downarrow$

$= 120x^3y^7$