Lesson 1.6 Exercises, pages 68–73

A

3. Determine whether each infinite geometric series has a finite sum. How do you know?

a) $2 + 3 + 4.5 + 6.75 + \ldots$

r is: $\frac{3}{2} = 1.5$, so the sum is not finite.

b) $-0.5 - 0.05 - 0.005 - 0.0005 - \ldots$

r is: $\frac{-0.05}{-0.5} = 0.1$, so the sum is finite.

c) $\frac{1}{2} - \frac{3}{8} + \frac{9}{32} - \frac{27}{128} + \ldots$

r is: $\dfrac{-\frac{3}{8}}{\frac{1}{2}} = -\frac{3}{4}$, so the sum is finite.

d) $0.1 + 0.2 + 0.4 + 0.8 + \ldots$

r is: $\frac{0.2}{0.1} = 2$, so the sum is not finite.

4. Write the first 4 terms of each infinite geometric series.

a) $t_1 = -4, r = 0.3$

$t_1 = -4$
t_2 is: $-4(0.3) = -1.2$
t_3 is: $-1.2(0.3) = -0.36$
t_4 is: $-0.36(0.3) = -0.108$

b) $t_1 = 1, r = -0.25$

$t_1 = 1$
t_2 is: $1(-0.25) = -0.25$
t_3 is: $-0.25(-0.25) = 0.0625$
t_4 is: $0.0625(-0.25) = -0.015\,625$

c) $t_1 = 4, r = \frac{1}{5}$

$t_1 = 4$
t_2 is: $4\left(\frac{1}{5}\right) = \frac{4}{5}$
t_3 is: $\frac{4}{5}\left(\frac{1}{5}\right) = \frac{4}{25}$
t_4 is: $\frac{4}{25}\left(\frac{1}{5}\right) = \frac{4}{125}$

d) $t_1 = -\frac{3}{2}, r = -\frac{3}{8}$

$t_1 = -\frac{3}{2}$
t_2 is: $-\frac{3}{2}\left(-\frac{3}{8}\right) = \frac{9}{16}$
t_3 is: $\frac{9}{16}\left(-\frac{3}{8}\right) = -\frac{27}{128}$
t_4 is: $-\frac{27}{128}\left(-\frac{3}{8}\right) = \frac{81}{1024}$

5. Each infinite geometric series converges. Determine each sum.

a) $8 + 2 + 0.5 + 0.125 + \ldots$

Use: $S_\infty = \frac{t_1}{1 - r}$

Substitute: $t_1 = 8$, $r = \frac{1}{4}$

$S_\infty = \frac{8}{1 - \frac{1}{4}}$

$S_\infty = 10.\overline{6}$

The sum is $10.\overline{6}$.

b) $-1 - \frac{3}{4} - \frac{9}{16} - \frac{27}{64} - \ldots$

Use: $S_\infty = \frac{t_1}{1 - r}$

Substitute: $t_1 = -1$, $r = \frac{3}{4}$

$S_\infty = \frac{-1}{1 - \frac{3}{4}}$

$S_\infty = -4$

The sum is -4.

c) $10 - \frac{20}{3} + \frac{40}{9} - \frac{80}{27} + \ldots$

Use: $S_\infty = \frac{t_1}{1 - r}$

Substitute: $t_1 = 10$, $r = -\frac{2}{3}$

$S_\infty = \frac{10}{1 - \left(-\frac{2}{3}\right)}$

$S_\infty = 6$

The sum is 6.

d) $-2 + \frac{2}{3} - \frac{2}{9} + \frac{2}{27} - \ldots$

Use: $S_\infty = \frac{t_1}{1 - r}$

Substitute: $t_1 = -2$, $r = -\frac{1}{3}$

$S_\infty = \frac{-2}{1 - \left(-\frac{1}{3}\right)}$

$S_\infty = -1.5$

The sum is -1.5.

B

6. What do you know about the common ratio of an infinite geometric series whose sum is finite?

The common ratio is less than 1 and greater than -1.

7. Use the given data about each infinite geometric series to determine the indicated value.

a) $t_1 = 21$, $S_\infty = 63$; determine r

Substitute for t_1 and S_∞ in $S_\infty = \frac{t_1}{1 - r}$.

$63 = \frac{21}{1 - r}$

$63 - 63r = 21$

$63r = 42$

$r = \frac{42}{63}$, or $\frac{2}{3}$

b) $r = -\frac{3}{4}$, $S_\infty = \frac{24}{7}$; determine t_1

Substitute for r and S_∞ in $S_\infty = \frac{t_1}{1 - r}$.

$\frac{24}{7} = \frac{t_1}{1 - \left(-\frac{3}{4}\right)}$

$\frac{24}{7}\left(\frac{7}{4}\right) = t_1$

$t_1 = 6$

8. Use an infinite geometric series to express each repeating decimal as a fraction.

a) $0.4\overline{97}$

$0.4\overline{97} = 0.4 + 0.097 + 0.000\,97 + \ldots$

This is an infinite geometric series with $t_1 = \frac{97}{1000}$ and r is $\frac{0.000\,97}{0.097} = \frac{1}{100}$

Substitute for t_1 and r in $S_\infty = \frac{t_1}{1 - r}$.

$$S_\infty = \frac{\frac{97}{1000}}{1 - \frac{1}{100}}$$

$$S_\infty = \frac{\frac{97}{1000}}{\frac{99}{100}}, \text{ or } \frac{97}{990}$$

Add: $\frac{4}{10} + \frac{97}{990} = \frac{493}{990}$

So, $0.4\overline{97} = \frac{493}{990}$

b) $1.\overline{143}$

$1.\overline{143} = 1 + 0.143 + 0.000\,143 + \ldots$

This is an infinite geometric series with $t_1 = \frac{143}{1000}$ and r is $\frac{0.000\,143}{0.143} = \frac{1}{1000}$

Substitute for t_1 and r in $S_\infty = \frac{t_1}{1 - r}$.

$$S_\infty = \frac{\frac{143}{1000}}{1 - \frac{1}{1000}}$$

$$S_\infty = \frac{\frac{143}{1000}}{\frac{999}{1000}}, \text{ or } \frac{143}{999}$$

Add: $1 + \frac{143}{999} = \frac{1142}{999}$

So, $1.\overline{143} = \frac{1142}{999}$

9. The hour hand on a clock is pointing to 12. The hand is rotated clockwise 180°, then another 60°, then another 20°, and so on. This pattern continues.

a) Which number would the hour hand approach if this rotation continued indefinitely? Explain what you did.

The angles, in degrees, that the hand rotates through form a geometric sequence with $t_1 = 180$ and $r = \frac{1}{3}$. The total angle turned through is the related infinite geometric series:

$$180 + \frac{180}{3} + \frac{180}{3^2} + \frac{180}{3^3} + \ldots$$

Use: $S_\infty = \frac{t_1}{1 - r}$ Substitute: $t_1 = 180$, $r = \frac{1}{3}$

$$S_\infty = \frac{180}{1 - \frac{1}{3}}$$

$$S_\infty = 270$$

When the hour hand has rotated 270° clockwise from 12, it will point to 9. So, if the rotation continued indefinitely, the hour hand would approach 9.

b) What assumptions did you make?

I assumed that the angle measures formed an infinite geometric series that converged.

10. Brad has a balance of \$500 in a bank account. Each month he spends 40% of the balance remaining in the account.

a) Express the total amount Brad spends in the first 4 months as a series. Is the series geometric? Explain.

After:	Amount spent	Amount remaining
1 month	$\$500(0.4) = \$200$	$\$500(0.6) = \$300$
2 months	$\$300(0.4) = \$120$, or $\$500(0.6)(0.4)$	$\$300(0.6) = \$180$, or $\$500(0.6)^2$
3 months	$\$180(0.4) = \$72$, or $\$500(0.6)^2(0.4)$	$\$180(0.6) = \$108$, or $\$500(0.6)^3$
4 months	$\$108(0.4) = \$43.20$, or $\$500(0.6)^3(0.4)$	

The amounts spent are:
$\$500(0.4) + \$500(0.6)(0.4) + \$500(0.6)^2(0.4) + \$500(0.6)^3(0.4)$
This is a geometric series with $t_1 = \$500(0.4)$ and $r = 0.6$

b) Determine the approximate amount Brad spends in 10 months. Explain what you did.

The amount Brad spends in 10 months is the sum of the first 10 terms of the series in part a.
Use: $S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1$
Substitute: $n = 10$, $t_1 = 200$, $r = 0.6$
$S_{10} = \frac{200(1 - 0.6^{10})}{1 - 0.6}$
$S_{10} = 496.9766...$
Brad spends about \$496.98 in 10 months.

c) Suppose Brad could continue this pattern of spending indefinitely. Would he eventually empty his bank account? Explain.

No, because Brad can only spend money in dollars and cents, and not fractions of a cent, so each amount he spends will be rounded to the nearest cent. Continuing the pattern of spending 40% each month, and rounding to the nearest cent each time, Brad will eventually end up with \$0.01 remaining in his account. Since 40% of \$0.01 is less than 1 penny, this amount will never be spent.

C

11. Write the product of $0.\overline{a}$ and $0.\overline{b}$ as a fraction, where a and b represent 1-digit natural numbers. Explain your strategy.

$0.\overline{a} = 0.a + 0.0a + 0.00a + \ldots$

This is an infinite geometric series with $t_1 = 0.a$, or $\frac{a}{10}$ and $r = 0.1$, or $\frac{1}{10}$.

Use: $S_\infty = \frac{t_1}{1 - r}$ Substitute: $t_1 = \frac{a}{10}$, $r = \frac{1}{10}$

$$S_\infty = \frac{\frac{a}{10}}{1 - \left(\frac{1}{10}\right)}$$

$$S_\infty = \frac{\frac{a}{10}}{\frac{9}{10}}$$

$$S_\infty = \frac{a}{9}$$

So, $0.\overline{a} = \frac{a}{9}$; similarly, $0.\overline{b} = \frac{b}{9}$; then $(0.\overline{a})(0.\overline{b})$ is

$$\left(\frac{a}{9}\right)\left(\frac{b}{9}\right) = \frac{ab}{81}$$

12. Create 2 different infinite geometric series with a sum of 4. Explain what you did.

Sample response: Choose a value for r between -1 and 1, then determine a value for t_1.

Let $r = -0.25$.

Use: $S_\infty = \frac{t_1}{1 - r}$

Substitute:

$S_\infty = 4$, $r = -0.25$

$$4 = \frac{t_1}{1 - (-0.25)}$$

$t_1 = 5$

Let $r = 0.6$.

Use: $S_\infty = \frac{t_1}{1 - r}$

Substitute:

$S_\infty = 4$, $r = 0.6$

$$4 = \frac{t_1}{1 - 0.6}$$

$t_1 = 1.6$

One series is: $5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \ldots$

Another series is: $1.6 + 0.96 + 0.576 + 0.3456 + \ldots$

13. Determine the sum of this infinite geometric series:
$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{32}} + \frac{1}{\sqrt{128}} + \frac{1}{\sqrt{512}} + \ldots$$

The common ratio is r: $\dfrac{\frac{1}{\sqrt{8}}}{\frac{1}{\sqrt{2}}} = \dfrac{1}{\sqrt{4}}$, or $\dfrac{1}{2}$

Use: $S_\infty = \dfrac{t_1}{1 - r}$ Substitute: $t_1 = \dfrac{1}{\sqrt{2}}$, $r = \dfrac{1}{2}$

$$S_\infty = \frac{\frac{1}{\sqrt{2}}}{1 - \frac{1}{2}}$$

$$S_\infty = \frac{2}{\sqrt{2}}$$

The sum is $\dfrac{2}{\sqrt{2}}$.