Lesson 4.4 Exercises, pages 314–321

A

3. For each function below, determine possible functions f and g so that $y = f(g(x))$.

a) $y = (x + 4)^2$

Sample solution:
Let $f(g(x)) = (x + 4)^2$
Replace $x + 4$ with x.
Then, $g(x) = x + 4$ and $f(x) = x^2$

b) $y = \sqrt{x + 5}$

Sample solution:
Let $f(g(x)) = \sqrt{x + 5}$
Replace $x + 5$ with x.
Then, $g(x) = x + 5$ and $f(x) = \sqrt{x}$

c) $y = \frac{1}{x - 2}$

Sample solution:
Let $f(g(x)) = \frac{1}{x - 2}$
Replace $x - 2$ with x.
Then, $g(x) = x - 2$ and $f(x) = \frac{1}{x}$

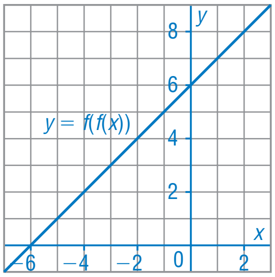
d) $y = (6 - x)^3$

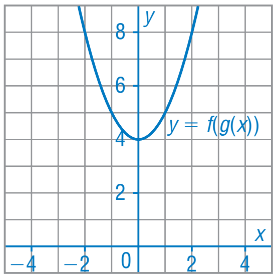
Sample solution:
Let $f(g(x)) = (6 - x)^3$
Replace $6 - x$ with x.
Then, $g(x) = 6 - x$ and $f(x) = x^3$

B

4. Given $f(x) = x + 3$ and $g(x) = x^2 + 1$, sketch the graph of each composite function below then state its domain and range.

a) $y = f(f(x))$ **b)** $y = f(g(x))$





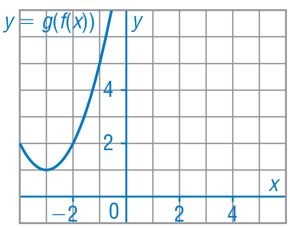
Make a table of values for the functions.

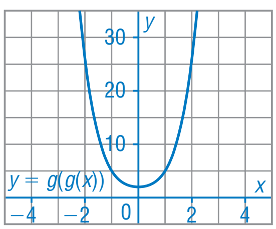
x	$f(x)$	$f(f(x))$	$g(x)$	$f(g(x))$	$g(f(x))$	$g(g(x))$
−4	−1	2	17	20	2	290
−3	0	3	10	13	1	101
−2	1	4	5	8	2	26
−1	2	5	2	5	5	5
0	3	6	1	4	10	2
1	4	7	2	5	17	5
2	5	8	5	8	26	26

a) Graph the points with coordinates $(x, f(f(x)))$ that fit on the grid.
Draw a line through the points for the graph of $y = f(f(x))$.
From the graph, the domain is $x \in \mathbb{R}$ and the range is $y \in \mathbb{R}$.

b) Graph the points with coordinates $(x, f(g(x)))$ that fit on the grid.
Draw a smooth curve through the points for the graph of $y = f(g(x))$.
From the graph, the domain is $x \in \mathbb{R}$ and the range is $y \geq 4$.

c) $y = g(f(x))$ **d)** $y = g(g(x))$





c) Graph the points with coordinates $(x, g(f(x)))$ that fit on the grid.
Draw a smooth curve through the points for the graph of $y = g(f(x))$.
From the graph, the domain is $x \in \mathbb{R}$. From the table, the range is $y \geq 1$.

d) Graph the points with coordinates $(x, g(g(x)))$ that fit on the grid.
Draw a smooth curve through the points for the graph of $y = g(g(x))$.
From the graph, the domain is $x \in \mathbb{R}$. From the table, the range is $y \geq 2$.

5. Consider the function $h(x) = (x - 1)(x + 5)$.

a) Why is it incorrect to write $h(x) = f(g(x))$, where $f(x) = x - 1$ and $g(x) = x + 5$?

It is incorrect because, as written, $h(x)$ is the product of $f(x)$ and $g(x)$, not their composition.

b) For what functions $f(x)$ and $g(x)$ is $h(x)$ a composite function?

Expand: $h(x) = (x - 1)(x + 5)$
$h(x) = x^2 + 4x - 5$
Complete the square: $h(x) = (x^2 + 4x + 4) - 9$
$h(x) = (x + 2)^2 - 9$
Possible functions are: $f(x) = x^2 - 9$ **and** $g(x) = x + 2$ **for** $h(x) = f(g(x))$

6. For each pair of functions below:

i) Determine an explicit equation for the indicated composite function.

ii) State the domain of the composite function, and explain any restrictions on the variable.

a) $f(x) = \sqrt{x + 1}$ and $g(x) = x^2 - x - 6$; $g(f(x))$

i) In $g(x) = x^2 - x - 6$, **replace** x **with** $\sqrt{x + 1}$.
$g(f(x)) = (\sqrt{x + 1})^2 - \sqrt{x + 1} - 6$
$g(f(x)) = x + 1 - \sqrt{x + 1} - 6$
$g(f(x)) = x - 5 - \sqrt{x + 1}$

ii) The domain of $f(x) = \sqrt{x + 1}$ **is** $x \geq -1$.
The domain of $g(x) = x^2 - x - 6$ **is** $x \in \mathbb{R}$.
So, the domain of $g(f(x))$ **is** $x \geq -1$.
The variable x is restricted because the square root of a real number is only defined for numbers that are greater than or equal to 0.

b) $f(x) = \sqrt{x - 1}$ and $g(x) = \frac{1}{x + 3}$; $g(f(x))$

i) In $g(x) = \frac{1}{x + 3}$, **replace** x **with** $\sqrt{x - 1}$.

$g(f(x)) = \frac{1}{\sqrt{x - 1} + 3}$

ii) The domain of $f(x) = \sqrt{x - 1}$ **is** $x \geq 1$.

The domain of $g(x) = \frac{1}{x + 3}$ **is** $x \neq -3$.

-3 is not in the range of $f(x)$.

So, the domain of $g(f(x))$ **is** $x \geq 1$.
The variable x is restricted because the square root of a real number is only defined for numbers that are greater than or equal to 0.

c) $f(x) = \sqrt{x + 3}$ and $g(x) = 2x - 1$; $f(g(x))$

i) In $f(x) = \sqrt{x + 3}$, replace x with $2x - 1$.
$f(g(x)) = \sqrt{2x - 1 + 3}$
$f(g(x)) = \sqrt{2x + 2}$

ii) The domain of $g(x) = 2x - 1$ is $x \in \mathbb{R}$.
The domain of $f(x) = \sqrt{x + 3}$ is $x \geq -3$.
So, $g(x) \geq -3$
$2x - 1 \geq -3$
$2x \geq -2$
$x \geq -1$
So, the domain of $f(g(x))$ is $x \geq -1$.
The variable x is restricted because the square root of a real number is only defined for numbers that are greater than or equal to 0.

d) $f(x) = \frac{1}{x - 1}$ and $g(x) = x^2 + 2x$; $f(f(x))$

i) In $f(x) = \frac{1}{x - 1}$, replace x with $\frac{1}{x - 1}$.
$f(f(x)) = \frac{1}{\frac{1}{x - 1} - 1}$, which simplifies to $f(f(x)) = \frac{x - 1}{2 - x}$, $x \neq 1$

ii) The domain of $f(x) = \frac{1}{x - 1}$ is $x \neq 1$.
Also, $2 - x \neq 0$
$x \neq 2$
So, the domain of $f(f(x))$ is $x \neq 1$ and $x \neq 2$.
The variable x is restricted because the denominator of a fraction can never be 0.

7. For each function below
i) Determine possible functions f and g so that $y = f(g(x))$.
ii) Determine possible functions f, g, and h so that $y = f(g(h(x)))$.

a) $y = x^2 - 6x + 5$

Sample solution:
$y = x^2 - 6x + 5$
$y = (x^2 - 6x + 9) - 4$
$y = (x - 3)^2 - 4$
Let $f(g(x)) = (x - 3)^2 - 4$
i) Replace $x - 3$ with x.
Then, $g(x) = x - 3$ and
$f(x) = x^2 - 4$
ii) Replace $x - 3$ with x.
Then, $h(x) = x - 3$, $g(x) = x^2$,
and $f(x) = x - 4$

b) $y = -3x^2 - 30x - 40$

Sample solution:
$y = -3x^2 - 30x - 40$
$y = -3(x^2 + 10x + 25) + 75 - 40$
$y = -3(x + 5)^2 + 35$
Let $f(g(x)) = -3(x + 5)^2 + 35$
i) Replace $x + 5$ with x.
Then, $g(x) = x + 5$ and
$f(x) = -3x^2 + 35$
ii) Replace $x + 5$ with x.
Then, $h(x) = x + 5$, $g(x) = x^2$,
and $f(x) = -3x + 35$

c) $y = \sqrt{(x - 2)^2 + 3}$

Sample solution:
Let $f(g(x)) = \sqrt{(x - 2)^2 + 3}$
i) Replace $x - 2$ with x.
Then, $g(x) = x - 2$ and $f(x) = \sqrt{x^2 + 3}$
ii) Replace $x - 2$ with x.
Then, $h(x) = x - 2$, $g(x) = x^2$, and $f(x) = \sqrt{x + 3}$

d) $y = \sqrt{x^2 + 4x + 3}$

Sample solution:
$y = \sqrt{x^2 + 4x + 3}$
$y = \sqrt{(x^2 + 4x + 4) - 1}$
$y = \sqrt{(x + 2)^2 - 1}$
Let $f(g(x)) = \sqrt{(x + 2)^2 - 1}$
i) Replace $x + 2$ with x.
Then, $g(x) = x + 2$ and $f(x) = \sqrt{x^2 - 1}$
ii) Replace $x + 2$ with x.
Then, $h(x) = x + 2$, $g(x) = x^2$, and $f(x) = \sqrt{x - 1}$

8. Create composite functions using either or both functions in each pair of functions below. In each case, how many different composite functions could you create? Justify your answer.

a) $f(x) = |x|$ and $g(x) = \frac{1}{x}$

$f(f(x)) = ||x||$, which simplifies to $f(f(x)) = |x|$

$f(g(x)) = \left|\frac{1}{x}\right|$, which simplifies to $f(g(x)) = \frac{1}{|x|}$

$g(f(x)) = \frac{1}{|x|}$

$g(g(x)) = \frac{1}{\frac{1}{x}}$, which simplifies to $g(g(x)) = x, x \neq 0$

There are only 3 different composite functions, because $f(g(x)) = g(f(x))$.

b) $f(x) = \sqrt{x}$ and $g(x) = |x|$

$f(f(x)) = \sqrt{\sqrt{x}}$
$f(g(x)) = \sqrt{|x|}$
$g(f(x)) = |\sqrt{x}|$, which simplifies to $g(f(x)) = \sqrt{x}$
$g(g(x)) = ||x||$, which simplifies to $g(g(x)) = |x|$
There are 4 different composite functions.

c) $f(x) = x^3$ and $g(x) = \frac{1}{x}$

$f(f(x)) = (x^3)^3$, which simplifies to $f(f(x)) = x^9$

$f(g(x)) = \left(\frac{1}{x}\right)^3$, which simplifies to $f(g(x)) = \frac{1}{x^3}$

$g(f(x)) = \frac{1}{x^3}$

$g(g(x)) = \frac{1}{\frac{1}{x}}$, which simplifies to $g(g(x)) = x, x \neq 0$

There are only 3 different composite functions, because $f(g(x)) = g(f(x))$.

9. Given the function $y = \frac{x}{\sqrt{x-3}}$, determine possible functions:

a) f and g so that $y = \frac{f(x)}{g(x)}$

Sample solution:
$f(x) = x$ and $g(x) = \sqrt{x-3}$

b) f, g, and h so that $y = \frac{f(x)}{g(h(x))}$

Sample solution:
Replace $x - 3$ with x.
Let $h(x) = x - 3$, then $g(x) = \sqrt{x}$, and $f(x) = x$.

c) f and g so that $y = f(g(x))$

Sample solution:
When $g(x)$ replaces x in $f(x)$, the numerator must be x and the denominator must be $\sqrt{x-3}$. So, $g(x) = x - 3$ and $f(x) = \frac{x+3}{\sqrt{x}}$

10. Given the functions $f(x) = \sqrt{x}$, $g(x) = x^2 - x + 6$, and $k(x) = \frac{2}{x}$, write an explicit equation for each combination.

a) $h(x) = f(g(x)) + k(x)$

For $f(g(x))$, replace x in $f(x) = \sqrt{x}$ with $x^2 - x + 6$.
Then, $f(g(x)) = \sqrt{x^2 - x + 6}$
So, $h(x) = \sqrt{x^2 - x + 6} + \frac{2}{x}$, $x \neq 0$

b) $h(x) = g(f(x)) - f(g(x))$

For $g(f(x))$, replace x in $g(x) = x^2 - x + 6$ with $\sqrt{x}$.
Then, $g(f(x)) = (\sqrt{x})^2 - \sqrt{x} + 6$
Or, $g(f(x)) = x - \sqrt{x} + 6$, $x \geq 0$
So, $h(x) = x - \sqrt{x} + 6 - \sqrt{x^2 - x + 6}$, $x \geq 0$

c) $h(x) = k(g(x)) + k(f(x))$

For $k(g(x))$, replace x in $k(x) = \frac{2}{x}$ with $x^2 - x + 6$.
Then, $k(g(x)) = \frac{2}{x^2 - x + 6}$
For $k(f(x))$, replace x in $k(x) = \frac{2}{x}$ with $f(x) = \sqrt{x}$
Then, $k(f(x)) = \frac{2}{\sqrt{x}}$, $x > 0$
So, $h(x) = \frac{2}{x^2 - x + 6} + \frac{2}{\sqrt{x}}$, $x > 0$

d) $h(x) = f(g(x)) \cdot k(x)$

From part a,
$f(g(x)) = \sqrt{x^2 - x + 6}$
So, $h(x) = \sqrt{x^2 - x + 6} \cdot \left(\frac{2}{x}\right)$, $x \neq 0$

11. Given the function $y = (x^2 - 9)\sqrt{x + 2}$, determine possible functions in each case:

a) functions f and g so that $y = f(x) \cdot g(x)$

Sample solution:
$f(x) = x^2 - 9$ and $g(x) = \sqrt{x + 2}$

b) functions f, g, and h so that $y = f(x) \cdot g(h(x))$

Sample solution:
$f(x) = x^2 - 9$
For $g(h(x))$, let $h(x) = x + 2$, then $g(x) = \sqrt{x}$

c) functions f, g, h, and k so that $y = f(x) \cdot k(x) \cdot g(h(x))$

Sample solution:
From part b, for $g(h(x))$, let $h(x) = x + 2$, then $g(x) = \sqrt{x}$
Factor: $x^2 - 9 = (x + 3)(x - 3)$
Then, $f(x) = x + 3$ and $k(x) = x - 3$

12. Is there a function $f(x)$ such that each relationship is true? Justify your answer.

a) $f(f(x)) = f(x)$

Yes, when $f(x) = x$, then $f(f(x)) = x$

b) $f(f(x)) = f(x) + f(x)$

Yes, when $f(x) = 2x$, then $f(f(x)) = 4x$ and $f(x) + f(x) = 2x + 2x$, or $4x$

C

13. Given $f(x) = \frac{1}{x - 2}$, $g(x)$ is a quadratic function, and $h(x) = f(g(x))$, determine an explicit equation for $g(x)$ for each situation below. Explain your strategies.

a) The domain of $h(x)$ is $x \in \mathbb{R}$.

Sample solution: The denominator of $h(x)$ must never be 0.
When $g(x) = x^2 + 3$, then $f(g(x)) = \frac{1}{x^2 + 3 - 2}$, which simplifies to
$f(g(x)) = \frac{1}{x^2 + 1}$.

b) The domain of $h(x)$ is $x \neq a$ and $x \neq b$, where a and b are real numbers.

Sample solution: There must be exactly two values of x that make the denominator of $h(x)$ equal to 0. When $g(x) = x^2 + 1$, then
$f(g(x)) = \frac{1}{x^2 + 1 - 2}$, which simplifies to $f(g(x)) = \frac{1}{x^2 - 1}$.
So, $a = 1$ and $b = -1$

c) The domain of $h(x)$ is $x \neq c$, where c is a real number.

Sample solution: There must be exactly one value of x that makes the denominator of $h(x)$ equal to 0. When $g(x) = x^2 + 2$, then $f(g(x)) = \dfrac{1}{x^2 + 2 - 2}$, which simplifies to $f(g(x)) = \dfrac{1}{x^2}$. So, $c = 0$

14. Use $f(x) = \dfrac{1 - x}{1 + x}$.

a) Determine an explicit equation for $f(f(x))$, then state the domain of the function.

In $f(x) = \dfrac{1 - x}{1 + x}$, replace x with $\dfrac{1 - x}{1 + x}$

$$f(f(x)) = \frac{1 - \frac{1 - x}{1 + x}}{1 + \frac{1 - x}{1 + x}}$$

$$= \frac{\frac{1 + x - (1 - x)}{1 + x}}{\frac{1 + x + (1 - x)}{1 + x}}$$

$$= x, x \neq -1$$

The domain of the function is: $x \neq -1$

b) What is the inverse of $f(x)$? Explain.

Since $f(f(x)) = x, x \neq -1$, then $f(x)$ is its own inverse.
So, the inverse of $f(x)$ is $f^{-1}(x) = \dfrac{1 - x}{1 + x}$.