Lesson 6.7 Exercises, pages 548–555

A

3. Identify the transformations that would be applied to the graph of
$y = \cos x$ to get the graph of $y = \frac{3}{4} \cos \frac{\pi}{5}(x + 3) - \frac{3}{2}$.

Compare $y = \frac{3}{4} \cos \frac{\pi}{5}(x + 3) - \frac{3}{2}$ with $y = a \cos b(x - c) + d$:

$a = \frac{3}{4}$, so the graph of $y = \cos x$ is compressed vertically by a factor of $\frac{3}{4}$.

$b = \frac{\pi}{5}$, so the graph of $y = \cos x$ is stretched horizontally by a factor of $\frac{5}{\pi}$.

$c = -3$, so the graph of $y = \cos x$ is translated 3 units left.

$d = -\frac{3}{2}$, so the graph of $y = \cos x$ is translated $\frac{3}{2}$ units down.

4. Identify the following characteristics of each graph below:
amplitude; period; phase shift; equation of the centre line; domain;
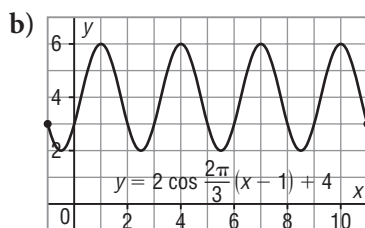maximum value; minimum value; range

a)
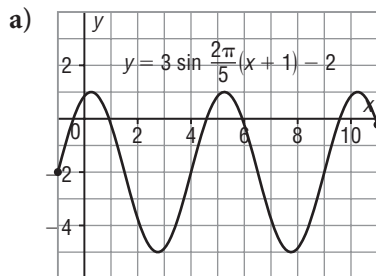
The amplitude is 3. The period is $\dfrac{2\pi}{\frac{2\pi}{5}} = 5$. A possible phase shift is -1.

The equation of the centre line is $y = -2$. The domain is $-1 \le x \le 11$.
The maximum value is 1. The minimum value is -5.
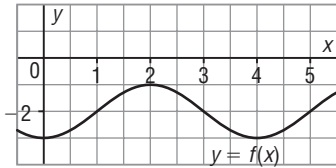The range is $-5 \le y \le 1$.

b)

The amplitude is 2. The period is $\dfrac{2\pi}{\frac{2\pi}{3}} = 3$. A possible phase shift is 1.

The equation of the centre line is $y = 4$. The domain is $-1 \le x \le 11$.
The maximum value is 6. The minimum value is 2. The range is $2 \le y \le 6$.

5. a) For the function graphed below, identify the values of *a*, *b*, *c*, and *d* in $y = a \sin b(x - c) + d$, then write an equation for the function.

Sample response: The equation of the centre line is $y = -2$, so the vertical translation is 2 units down and $d = -2$.

The amplitude is: $\dfrac{-1 - (-3)}{2} = 1$, so $a = 1$

Choose the *x*-coordinates of two adjacent minimum points, such as 0 and 4. The period is: $4 - 0 = 4$
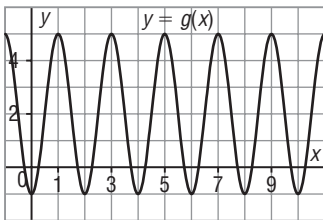
So, *b* is: $\dfrac{2\pi}{4} = \dfrac{\pi}{2}$

A possible phase shift is 1, and $c = 1$.

Substitute for *a*, *b*, *c*, and *d* in: $y = a \sin b(x - c) + d$

An equation is: $y = \sin \dfrac{\pi}{2}(x - 1) - 2$

b) For the function graphed below, identify the values of *a*, *b*, *c*, and *d* in $y = a \cos b(x - c) + d$, then write an equation for the function.

Sample response: The equation of the centre line is $y = 2$, so the vertical translation is 2 units up and $d = 2$.

The amplitude is: $\dfrac{5 - (-1)}{2} = 3$, so $a = 3$

Choose the *x*-coordinates of two adjacent maximum points, such as 1 and 3. The period is: $3 - 1 = 2$

So, *b* is: $\dfrac{2\pi}{2} = \pi$

A possible phase shift is 1, and $c = 1$.

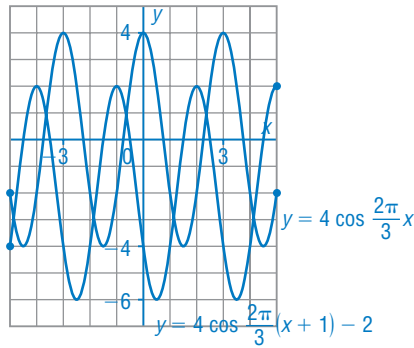**Substitute for *a*, *b*, *c*, and *d* in: $y = a \cos b(x - c) + d$**

An equation is: $y = 3 \cos \pi(x - 1) + 2$

6. Use transformations to sketch a graph of each function for
$-5 \leq x \leq 5$.

a) $y = 4 \cos \frac{2\pi}{3}(x + 1) - 2$

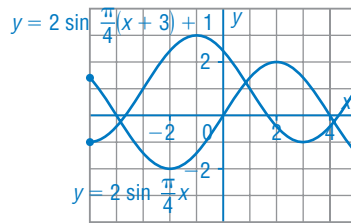The period of the graph is $\frac{2\pi}{\frac{2\pi}{3}} = 3$.

Sketch the graph of $y = 4 \cos \frac{2\pi}{3}x$,

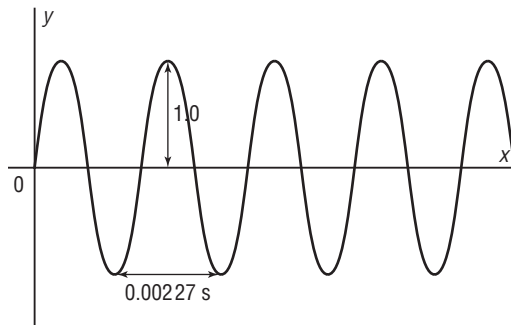then translate the graph 1 unit left and 2 units down for the graph of $y = 4 \cos \frac{2\pi}{3}(x + 1) - 2$.

b) $y = 2 \sin \frac{\pi}{4}(x + 3) + 1$

The period of the graph is $\frac{2\pi}{\frac{\pi}{4}} = 8$.

Sketch the graph of $y = 2 \sin \frac{\pi}{4}x$,

then translate the graph 3 units left and 1 unit up for the graph of $y = 2 \sin \frac{\pi}{4}(x + 3) + 1$.

7. A sound wave is a sinusoidal curve, as illustrated by the oscilloscope image below. Determine the function that best describes this graph.

The graph is a sine curve with amplitude 1 and period 0.002 27 s.
An equation of the function has the form: $y = a \sin bx$

Substitute: $a = 1$ and $b = \frac{2\pi}{0.002\ 27}$

A function is: $y = \sin \frac{2\pi x}{0.002\ 27}$

8. A vertical wheel with radius 50 cm rotates about an axle that is 60 cm above the ground. A marker is placed at the top of the wheel. The wheel completes one rotation every 4 s.

a) i) Explain why a cosine function would be an appropriate model for the height, h centimetres, of the marker at any time t seconds.

Assume the marker is at the top of the wheel at time $t = 0$. Then the graph of the height of the marker against time is a sinusoidal curve with its first maximum point on the vertical axis, so its equation can be represented by a cosine function.

ii) For the graph of the cosine function from part i, identify the: period; phase shift; equation of the centre line; and amplitude. Explain how each characteristic relates to the conditions in the problem.

The period is the time in seconds for one revolution, which is 4.
The phase shift is 0, because the motion begins when the marker is at the maximum height.
The constant in the equation of the centre line is the distance of the axle above the ground, which is 60. So, the equation is: $h = 60$
The amplitude is the radius of the wheel, which is 50.

b) Write an equation of a cosine function that models the motion of the wheel.

An equation has the form: $h = a \cos b(t - c) + d$
Substitute: $a = 50; b = \frac{2\pi}{4}$, or $\frac{\pi}{2}; c = 0; d = 60$
A possible equation is: $h = 50 \cos \frac{\pi}{2}t + 60$

9. The Fisheries and Oceans Canada website provided the following data for tide heights at Rankin Inlet on May 9, 2011.
Low tide: 1.0 m at 02:09; and 0.9 m at 14:28
High tide: 3.6 m at 08:09; and 3.7 m at 20:49
Assume the tide heights can be modelled using a sinusoidal function.

a) Write an equation of a sinusoidal function that models these data.

Sample response:
Graph the data. Let the time after midnight be t hours and the height be h metres.
Convert the times to decimals of an hour, then plot the points.

02:09 is $2\frac{9}{60}$ h $= 2.15$ h

14:28 is $14\frac{28}{60}$ h $\doteq 14.47$ h

08:09 h is $8\frac{9}{60}$ h $= 8.15$ h

20:49 h is $20\frac{49}{60}$ h $\doteq 20.82$ h

Tide Heights for Rankin Inlet, May 9, 2011

Points plotted: (8.15, 3.6); (20.82, 3.7); (2.15, 1); (14.47, 0.9)

An equation has the form:
$h = a \cos b(t - c) + d$
The first maximum has approximate coordinates: (8.15, 3.6)
The first minimum has approximate coordinates: (2.15, 1.0)

So, the amplitude is approximately: $\frac{3.6 - 1.0}{2} = 1.3$; so $a \doteq 1.3$

The phase shift is approximately 8.15; so $c \doteq 8.15$

The period is approximately: 20:49 − 08:09 = 12:40, or $\frac{38}{3}$;

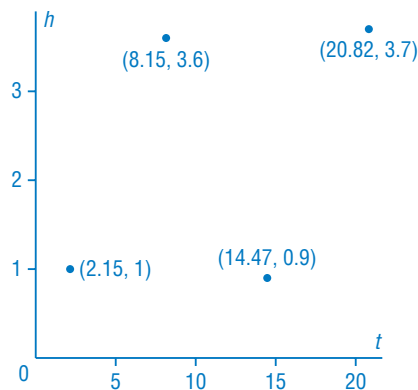so $b \doteq \dfrac{2\pi}{\frac{38}{3}}$, or $\dfrac{3\pi}{19}$

The equation of the centre line is approximately: $h = \frac{3.6 + 1.0}{2}$; or $h \doteq 2.3$; so $d \doteq 2.3$
Substitute the values of a, b, c, and d in the general equation above.

A possible equation is: $h = 1.3 \cos \frac{3\pi}{19}(t - 8.15) + 2.3$

b) Use the function to estimate the tide height at 09:00. Give the answer to the nearest half metre.

Use graphing technology. Graph: $y = 1.3 \cos \frac{3\pi}{19}(x - 8.15) + 2.3$
When $x = 9$, $y = 3.4861...$
So, the tide height at 09:00 is approximately 3.5 m.

10. These data show the fraction of the moon that was visible on alternate days in January, 2011.

a) Graph the data, then write an equation of a sinusoidal function that models these data.

Day	Visible fraction	Day	Visible fraction
01	0.11	17	0.91
03	0.01	19	0.99
05	0.01	21	0.97
07	0.08	23	0.85
09	0.21	25	0.65
11	0.38	27	0.43
13	0.57	29	0.23
15	0.76	31	0.08

Visible Fraction of the Moon
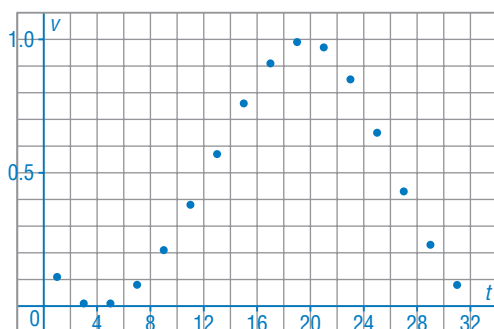
Sample solution: Let v represent the visible fraction of the moon and t represent the day of the month. Use a sine function to model the data:
$v(t) = a \sin b(t - c) + d$

The centre line has approximate equation $v = 0.5$, so $d \doteq 0.5$
This line intersects the graph at approximately $(12, 0.5)$, so the phase shift is approximately 12, and $c \doteq 12$

The amplitude is approximately 0.5, so $a \doteq 0.5$

The period is approximately 30 days, so $b \doteq \dfrac{2\pi}{30}$, or $\dfrac{\pi}{15}$
A function that approximates the data is: $v(t) = 0.5 \sin \dfrac{\pi}{15}(t - 12) + 0.5$

b) Use the table to identify the days when the maximum and minimum fractions of the moon were seen. Where are these points on the graph?

The maximum fraction of the moon was seen on day 19; this corresponds to the maximum point on the graph. The minimum fraction of the moon was seen on days 3 and 5; this is represented by the minimum point at day 4.

11. These data show the average hours of daylight for Vancouver, BC.

Date	Average daylight (h and min)
January 1	8:17
February 1	9:25
March 1	11:00
April 1	12:53
May 1	14:36
June 1	15:57
July 1	16:10
August 1	15:08
September 1	13:28
October 1	11:40
November 1	9:52
December 1	8:29

Average Hours of Daylight for Vancouver
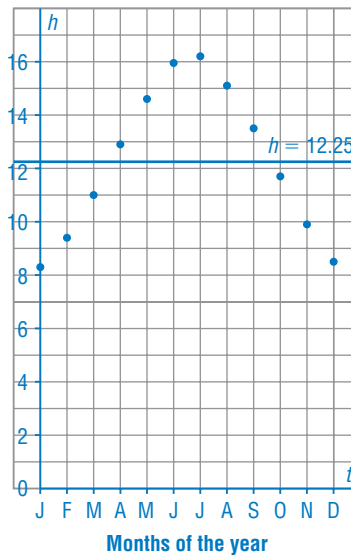
Months of the year

a) Graph the data, then write an equation of a sinusoidal function that models these data.

Sample solution: Let t represent the date and h represent the hours of daylight.

Assume the months have the same length.

Use a sine function to model the data:

$h(t) = a \sin b(t - c) + d$

There are 365 days in the year, so $0 \leq t \leq 365$.

The centre line has approximate equation

$h = \dfrac{8 \text{ h } 20 \text{ min } + 16 \text{ h } 10 \text{ min}}{2}$; or $h = 12 \text{ h } 15 \text{ min}$, so $d \doteq 12.25$

This line would intersect the graph approximately three-quarters of the way through March, so t is approximately $31 + 28 + 23$, or $t \doteq 82$

So, the phase shift is approximately 82, and $c \doteq 82$

The amplitude is approximately $\dfrac{16 \text{ h } 10 \text{ min } - 8 \text{ h } 20 \text{ min}}{2} \doteq 3 \text{ h } 55 \text{ min}$, so $a \doteq 3.9$

The period is approximately 365 days, so $b \doteq \dfrac{2\pi}{365}$

A function that approximates the data is:

$h(t) = 3.9 \sin \dfrac{2\pi}{365} (t - 82) + 12.25$

b) Use technology to graph the function. Use the graph and the equation.

  i) Estimate the average amount of daylight on April 15th.

Determine the day of the year for April 15th:
$31 + 28 + 31 + 15 = 105$
Graph: $Y = 3.9 \sin (2\pi(X - 82) \div 365) + 12.25$
Determine the value of Y when $X = 105$: $Y = 13.7540\ldots$
So, there are approximately 14 h of daylight on April 15th.
In the equation, substitute $t = 105$:

$h(105) = 3.9 \sin \dfrac{2\pi}{365} (105 - 82) + 12.25$

$\qquad = 13.7540\ldots$

So, there are approximately 14 h of daylight on April 15th.

  ii) On which dates are there 10 h of daylight?

On the screen for the graph in part i, graph $Y = 10$, then determine the X-coordinates of the points of intersection: $X = 46.2762\ldots$ and $X = 300.2237\ldots$
So, there are 10 h of daylight on day 46 (February 15th) and on day 300 (October 27th)
In the equation, substitute $h(t) = 10$:

$$10 = 3.9 \sin \dfrac{2\pi}{365} (t - 82) + 12.25 \qquad \text{Solve for } t.$$

$$-2.25 = 3.9 \sin \dfrac{2\pi}{365} (t - 82)$$

$$\dfrac{-2.25}{3.9} = \sin \dfrac{2\pi}{365} (t - 82)$$

$$\dfrac{2\pi}{365} (t - 82) = \sin^{-1}\left(\dfrac{-2.25}{3.9}\right)$$

$$= -0.6149\ldots$$

So, $\dfrac{2\pi}{365} (t - 82) = 2\pi - 0.6149\ldots$ or $\dfrac{2\pi}{365}(t - 82) = \pi + 0.6149\ldots$

$\dfrac{2\pi}{365} (t - 82) = 5.6682\ldots \qquad\qquad \dfrac{2\pi}{365}(t - 82) = 3.7565\ldots$

$t = 411.2762\ldots \qquad\qquad\qquad t = 300.2237\ldots$

Subtract 365 to get the 1st date: $\qquad t \doteq 300$, which is October 27th
$t \doteq 46$, which is February 15th

So, there are 10 h of daylight on approximately February 15th and October 27th.