

Lesson 7.2 Exercises, pages 592–599

A

Use algebra to solve each equation. Give exact values when possible; otherwise write the roots to the nearest degree or the nearest hundredth of a radian. Verify the solutions.

4. Solve each equation over the domain $0 \leq x < 2\pi$.

a) $\sin x = \frac{\sqrt{3}}{2}$

The reference angle is:

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

In Quadrant 1, $x = \frac{\pi}{3}$

In Quadrant 2, $x = \pi - \frac{\pi}{3}$,
or $\frac{2\pi}{3}$

b) $\tan x = \frac{1}{\sqrt{3}}$

The reference angle is:

$$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

In Quadrant 1, $x = \frac{\pi}{6}$

In Quadrant 3, $x = \pi + \frac{\pi}{6}$, or $\frac{7\pi}{6}$

5. Verify that each given value of x is a root of the equation.

a) $\tan^2 x - 3 = 0;$
 $x = \frac{\pi}{3}$

Substitute $x = \frac{\pi}{3}$ in each side of the equation.

$$\begin{aligned} \text{L.S.} &= \tan^2\left(\frac{\pi}{3}\right) - 3 \\ &= (\sqrt{3})^2 - 3 \\ &= 0 \\ &= \text{R.S.} \end{aligned}$$

Since the left side is equal to the right side, the root is verified.

b) $8 \sin^2 x + 6 \sin x + 1 = 0;$
 $x = \frac{7\pi}{6}$

Substitute $x = \frac{7\pi}{6}$ in each side of the equation.

$$\begin{aligned} \text{L.S.} &= 8 \sin^2\left(\frac{7\pi}{6}\right) + 6 \sin \frac{7\pi}{6} + 1 \\ &= 8\left(-\frac{1}{2}\right)^2 + 6\left(-\frac{1}{2}\right) + 1 \\ &= 2 - 3 + 1 \\ &= 0 \\ &= \text{R.S.} \end{aligned}$$

Since the left side is equal to the right side, the root is verified.

B

6. Solve each equation over the domain $0 \leq x < 2\pi$, then state the general solution.

a) $3 \cos x - 2 = 0$

$$\begin{aligned} 3 \cos x &= 2 \\ \cos x &= \frac{2}{3} \end{aligned}$$

In Quadrant 1,

$$x = \cos^{-1}\left(\frac{2}{3}\right)$$

$$x = 0.8410 \dots$$

In Quadrant 4,

$$x = 2\pi - 0.8410 \dots$$

$$x = 5.4421 \dots$$

The roots are: $x \doteq 0.84$ and

$$x \doteq 5.44$$

The general solution is:

$$x \doteq 0.84 + 2\pi k, k \in \mathbb{Z} \text{ or}$$

$$x \doteq 5.44 + 2\pi k, k \in \mathbb{Z}$$

b) $2 \tan x + \sqrt{5} = 0$

$$2 \tan x = -\sqrt{5}$$

$$\tan x = -\frac{\sqrt{5}}{2}$$

The reference angle is:

$$\tan^{-1}\left(\frac{\sqrt{5}}{2}\right) = 0.8410 \dots$$

In Quadrant 2,

$$x = \pi - 0.8410 \dots$$

$$x = 2.3005 \dots$$

In Quadrant 4,

$$x = 2\pi - 0.8410 \dots$$

$$x = 5.4421 \dots$$

The roots are: $x \doteq 2.30$ and

$$x \doteq 5.44$$

The general solution is:

$$x \doteq 2.30 + \pi k, k \in \mathbb{Z} \text{ or}$$

$$x \doteq 5.44 + \pi k, k \in \mathbb{Z}$$

7. Solve each equation for $-\pi \leq x \leq \pi$.

a) $3 \tan x - 3 = 5 \tan x - 1$ b) $5(1 + 2 \sin x) = 2 \sin x + 1$

$$2 \tan x = -2$$

$$\tan x = -1$$

The reference angle is:

$$\tan^{-1}(1) = \frac{\pi}{4}$$

In Quadrant 2,

$$x = \pi - \frac{\pi}{4}$$

$$x = \frac{3\pi}{4}$$

In Quadrant 4,

$$x = -\frac{\pi}{4}$$

The roots are: $x = \frac{3\pi}{4}$ and

$$x = -\frac{\pi}{4}$$

$$5 + 10 \sin x = 2 \sin x + 1$$

$$8 \sin x = -4$$

$$\sin x = -\frac{1}{2}$$

The reference angle is:

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

In Quadrant 3,

$$x = -\pi + \frac{\pi}{6}$$

$$x = -\frac{5\pi}{6}$$

In Quadrant 4,

$$x = -\frac{\pi}{6}$$

The roots are: $x = -\frac{\pi}{6}$ and

$$x = -\frac{5\pi}{6}$$

8. a) Solve each equation for $-180^\circ \leq x \leq 90^\circ$.

i) $2 \csc x = 6$

$$\csc x = 3$$

$$\sin x = \frac{1}{3}$$

The reference angle is:

$$\sin^{-1}\left(\frac{1}{3}\right) = 19.4712\dots^\circ$$

In Quadrant 1, $x = 19.4712\dots^\circ$

Quadrant 2 is not in the domain.

The root is: $x \doteq 19^\circ$

ii) $-6 = 3 \cot x$

$$-2 = \cot x$$

$$\tan x = -\frac{1}{2}$$

The reference angle is:

$$\tan^{-1}\left(\frac{1}{2}\right) = 26.5650\dots^\circ$$

Quadrant 2 is not in the domain.

In Quadrant 4, $x = -26.5650\dots^\circ$

The root is: $x \doteq -27^\circ$

b) Solve each equation for $-90^\circ \leq x \leq 180^\circ$.

i) $4 \sec x = -5$

$$\sec x = -\frac{5}{4}$$

$$\cos x = -\frac{4}{5}$$

The reference angle is:

$$\cos^{-1}\left(\frac{4}{5}\right) = 36.8698\dots^\circ$$

In Quadrant 2,

$$x = 180^\circ - 36.8698\dots^\circ$$

$$x = 143.1301\dots^\circ$$

There is no solution in Quadrant 3.

The root is: $x \doteq 143^\circ$

ii) $-\frac{1}{2} = \frac{1}{3} \csc x$

$$\csc x = -\frac{3}{2}$$

$$\sin x = -\frac{2}{3}$$

The reference angle is:

$$\sin^{-1}\left(\frac{2}{3}\right) = 41.8103\dots^\circ$$

There is no solution in Quadrant 3.

In Quadrant 4,

$$x = -41.8103\dots^\circ$$

The root is: $x \doteq -42^\circ$

9. For each equation, determine the general solution over the set of real numbers, then list the roots over the domain $-\pi \leq x < 0$.

a) $\cos 3x - 1 = 5 \cos 3x + 2$ b) $3 \sin 4x = 3 - 2 \sin 4x$

$$\begin{aligned} -3 &= 4 \cos 3x \\ \cos 3x &= -\frac{3}{4} \end{aligned}$$

$$3x = \cos^{-1}\left(-\frac{3}{4}\right)$$

The terminal arm of angle $3x$ lies in Quadrant 2 or 3.

The reference angle for angle $3x$ is:

$$\cos^{-1}\left(\frac{3}{4}\right) = 0.7227\dots$$

In Quadrant 2,

$$3x = \pi - 0.7227\dots$$

$$x = 0.8062\dots$$

In Quadrant 3,

$$3x = \pi + 0.7227\dots$$

$$x = 1.2881\dots$$

The period of $\cos 3x$ is $\frac{2\pi}{3}$, so the general solution is:

$$x \doteq 0.81 + \frac{2\pi}{3}k, k \in \mathbb{Z}, \text{ or}$$

$$x \doteq 1.29 + \frac{2\pi}{3}k, k \in \mathbb{Z}$$

In the given domain, the roots are:

$$x = 0.8062\dots - \frac{2\pi}{3}$$

$$x \doteq -1.29, \text{ and}$$

$$x = 1.2881\dots - \frac{2\pi}{3}$$

$$x \doteq -0.81, \text{ and}$$

$$x = 1.2881\dots - \frac{4\pi}{3}$$

$$x \doteq -2.90$$

$$5 \sin 4x = 3$$

$$\sin 4x = \frac{3}{5}$$

$$4x = \sin^{-1}\left(\frac{3}{5}\right)$$

The terminal arm of angle $4x$ lies in Quadrant 1 or 2.

The reference angle for angle $4x$ is:

$$\sin^{-1}\left(\frac{3}{5}\right) = 0.6435\dots$$

In Quadrant 1,

$$4x = 0.6435\dots$$

$$x = 0.1608\dots$$

In Quadrant 2,

$$4x = \pi - 0.6435\dots$$

$$x = 0.6245\dots$$

The period of $\sin 4x$ is $\frac{2\pi}{4}$, or $\frac{\pi}{2}$,

so the general solution is:

$$x \doteq 0.16 + \frac{\pi}{2}k, k \in \mathbb{Z}, \text{ or}$$

$$x \doteq 0.62 + \frac{\pi}{2}k, k \in \mathbb{Z}$$

In the given domain, the roots are:

$$x = 0.1608\dots - \frac{\pi}{2}$$

$$x \doteq -1.41, \text{ and}$$

$$x = 0.1608\dots - \pi$$

$$x \doteq -2.98, \text{ and}$$

$$x = 0.6245\dots - \frac{\pi}{2}$$

$$x \doteq -0.95, \text{ and}$$

$$x = 0.6245\dots - \pi$$

$$x \doteq -2.52$$

10. Two students determined the general solution of the equation $3 \sin x + 5 = 5(\sin x + 1)$. Joseph said the solution is $x = 2\pi k$ or $x = \pi + 2\pi k$, where k is an integer. Yeoun Sun said the solution is $x = \pi k$, where k is an integer. Who is correct? Explain.

Both students are correct because both expressions produce the same roots: $x = 0, x = \pm\pi, x = \pm 2\pi, x = \pm 3\pi$, and so on

11. Solve each equation over the domain $-\pi \leq x \leq \frac{\pi}{2}$.

a) $4 \cos^2 x - 3 = 0$

$$4 \cos^2 x = 3$$

$$\cos^2 x = \frac{3}{4}$$

$$\cos x = \pm \frac{\sqrt{3}}{2}$$

The reference angle is:

$$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

In Quadrant 1, $x = \frac{\pi}{6}$

In Quadrant 2, there is no solution in the given domain.

In Quadrant 3, $x = -\pi + \frac{\pi}{6}$

$$x = -\frac{5\pi}{6}$$

In Quadrant 4, $x = -\frac{\pi}{6}$

The roots are: $x = \pm \frac{\pi}{6}$ and

$$x = -\frac{5\pi}{6}$$

b) $2 \tan^2 x = 3$

$$\tan^2 x = \frac{3}{2}$$

$$\tan x = \pm \sqrt{\frac{3}{2}}$$

The reference angle is:

$$\tan^{-1}\left(\sqrt{\frac{3}{2}}\right) = 0.8860\dots$$

In Quadrant 1, $x = 0.8860\dots$

In Quadrant 2, there is no solution in the given domain.

In Quadrant 3, $x = -\pi + 0.8860\dots$

$$x = -2.2555\dots$$

In Quadrant 4, $x = -0.8860\dots$

The roots are: $x = \pm 0.89$ and

$$x = -2.26$$

12. Use factoring to solve each equation over the domain

$$-90^\circ \leq x < 270^\circ.$$

a) $2 \cos x \sin x - \cos x = 0$

$$(\cos x)(2 \sin x - 1) = 0$$

Either $\cos x = 0$

$$x = \pm 90^\circ$$

Or $2 \sin x - 1 = 0$

$$\sin x = 0.5$$

$$x = 30^\circ$$

or $x = 180^\circ - 30^\circ$

$$x = 150^\circ$$

The roots are: $x = 30^\circ$,

$x = \pm 90^\circ$, and $x = 150^\circ$

b) $3 \tan x + \tan^2 x = 2 \tan x$

$$\tan x + \tan^2 x = 0$$

$$(\tan x)(1 + \tan x) = 0$$

Either $\tan x = 0$

$$x = 0^\circ \text{ or } x = 180^\circ$$

Or $1 + \tan x = 0$

$$\tan x = -1$$

$$x = 135^\circ \text{ or } x = -45^\circ$$

The roots are: $x = 0^\circ$, $x = 180^\circ$,

$x = 135^\circ$, and $x = -45^\circ$

13. A student wrote the solution below to solve the equation

$2 \sin^2 x + \sin x = 1$ over the domain $0 \leq x < 2\pi$. Identify any errors, then write a correct solution.

$$2 \sin^2 x + \sin x = 1$$

$$(\sin x)(2 \sin x + 1) = 1$$

$$\sin x = 1 \quad \text{or} \quad 2 \sin x + 1 = 1$$

$$x = \frac{\pi}{2}$$

$$\sin x = 0$$

$$x = 0 \text{ or } x = \pi$$

$$2 \sin^2 x + \sin x - 1 = 0$$

$$(2 \sin x - 1)(\sin x + 1) = 0$$

Either $2 \sin x - 1 = 0$

$$\sin x = \frac{1}{2}$$

$$x = \frac{\pi}{6} \text{ or } x = \frac{5\pi}{6}$$

Or $\sin x + 1 = 0$

$$\sin x = -1$$

$$x = \frac{3\pi}{2}$$

There is an error in the second line of the solution. One side of the equation must be 0 before factoring, so collect all the terms on the left side.

14. Solve each equation over the domain $-2\pi \leq x \leq 2\pi$, then determine the general solution.

a) $2 \cos^2 x - \cos x - 1 = 0$

$$(2 \cos x + 1)(\cos x - 1) = 0$$

For $2 \cos x + 1 = 0$

$$2 \cos x = -1$$

$$\cos x = -\frac{1}{2}$$

The terminal arm of angle x lies in Quadrant 2 or 3.

The reference angle is:

$$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

In Quadrant 2,

$$x = \pi - \frac{\pi}{3}, \text{ or } \frac{2\pi}{3}$$

and, $x = -\pi - \frac{\pi}{3}, \text{ or } -\frac{4\pi}{3}$

In Quadrant 3,

$$x = \pi + \frac{\pi}{3}, \text{ or } \frac{4\pi}{3}$$

and, $x = -\pi + \frac{\pi}{3}, \text{ or } -\frac{2\pi}{3}$

For $\cos x - 1 = 0$

$$\cos x = 1$$

The terminal arm lies along the positive x -axis.

So, $x = \pm 2\pi$ or $x = 0$

The roots are: $x = 0, x = \pm \frac{2\pi}{3},$

$x = \pm \frac{4\pi}{3},$ and $x = \pm 2\pi$

Since consecutive roots differ by $\frac{2\pi}{3}$, the general solution is:

$$x = \frac{2\pi}{3}k, k \in \mathbb{Z}$$

b) $5 \sin^2 x + 3 \sin x = 2$

$$5 \sin^2 x + 3 \sin x - 2 = 0$$

$$(5 \sin x - 2)(\sin x + 1) = 0$$

For $5 \sin x - 2 = 0$

$$5 \sin x = 2$$

$$\sin x = \frac{2}{5}$$

The terminal arm of angle x lies in Quadrant 1 or 2.

The reference angle is:

$$\sin^{-1}\left(\frac{2}{5}\right) = 0.4115\dots$$

In Quadrant 1,

$$x = 0.4115\dots$$

and, $x = -2\pi + 0.4115\dots$

$$x = -5.8716\dots$$

In Quadrant 2,

$$x = \pi - 0.4115\dots$$

$$x = 2.7300\dots$$

and, $x = -\pi - 0.4115\dots$

$$x = -3.5531\dots$$

For $\sin x + 1 = 0$

$$\sin x = -1$$

The terminal arm lies along the negative y -axis.

So, $x = \frac{3\pi}{2}$ or $x = -\frac{\pi}{2}$

The roots are: $x \doteq -5.87,$

$x \doteq -3.55, x = -\frac{\pi}{2}, x \doteq 0.41,$

$x \doteq 2.73,$ and $x = \frac{3\pi}{2}$

Since the period is 2π , the general solution is:

$$x \doteq 0.41 + 2\pi k, k \in \mathbb{Z} \text{ or}$$

$$x \doteq 2.73 + 2\pi k, k \in \mathbb{Z} \text{ or}$$

$$x = \frac{3\pi}{2} + 2\pi k, k \in \mathbb{Z}$$

15. Solve each equation over the domain $0 \leq x < 2\pi$.

a) $4 \tan^2 x = 2 - 5 \tan x$

$$4 \tan^2 x + 5 \tan x - 2 = 0$$

$$\text{Use: } \tan x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{Substitute: } a = 4, b = 5, \\ c = -2$$

$$\tan x = \frac{-5 \pm \sqrt{5^2 - 4(4)(-2)}}{2(4)}$$

$$\tan x = \frac{-5 \pm \sqrt{57}}{8}$$

$$\text{For } \tan x = \frac{-5 + \sqrt{57}}{8}$$

$\tan x$ is positive when the terminal arm of angle x lies in Quadrant 1 or 3.

The reference angle is:

$$\tan^{-1}\left(\frac{-5 + \sqrt{57}}{8}\right) = 0.3085\dots$$

In Quadrant 1, $x = 0.3085\dots$

In Quadrant 3,
 $x = \pi + 0.3085\dots$

$x = 3.4501\dots$

$$\text{For } \tan x = \frac{-5 - \sqrt{57}}{8}$$

$\tan x$ is negative when the terminal arm of angle x lies in Quadrant 2 or 4.

The reference angle is:

$$\tan^{-1}\left(\frac{5 + \sqrt{57}}{8}\right) = 1.0032\dots$$

In Quadrant 2,

$x = \pi - 1.0032\dots$

$x = 2.1383\dots$

In Quadrant 4,

$x = 2\pi - 1.0032\dots$

$x = 5.2798\dots$

The roots are: $x \doteq 0.31$,

$x \doteq 2.14$, $x \doteq 3.45$, and

$x \doteq 5.28$

b) $4 \sin x + 3 = 2 \sin^2 x$

$$2 \sin^2 x - 4 \sin x - 3 = 0$$

$$\text{Use: } \sin x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{Substitute: } a = 2, b = -4, \\ c = -3$$

$$\sin x = \frac{4 \pm \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)}$$

$$\sin x = \frac{4 \pm \sqrt{40}}{4}, \text{ or } \frac{2 \pm \sqrt{10}}{2}$$

For $\sin x = \frac{2 + \sqrt{10}}{2}$; this is greater than 1, so there is no real solution.

$$\text{For } \sin x = \frac{2 - \sqrt{10}}{2}$$

$\sin x$ is negative when the terminal arm of angle x lies in Quadrant 3 or 4.

The reference angle is:

$$\sin^{-1}\left(\frac{\sqrt{10} - 2}{2}\right) = 0.6201\dots$$

In Quadrant 3,

$x = \pi + 0.6201\dots$

$x = 3.7617\dots$

In Quadrant 4,

$x = 2\pi - 0.6201\dots$

$x = 5.6630\dots$

The roots are: $x \doteq 3.76$ and

$x \doteq 5.66$

C

16. Write a second-degree trigonometric equation that has roots

$\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ over the domain $0 \leq x < 2\pi$.

Sample response: Since $\sin \frac{\pi}{6} = \sin \frac{5\pi}{6}$, these two roots can be determined from one factor, so use the sine ratio. Work backward.

$$\text{Either } x = \frac{\pi}{6} \quad \text{or} \quad x = \frac{\pi}{2}$$

$$\text{So, } \sin x = \frac{1}{2} \quad \text{or} \quad \sin x = 1$$

$$\text{Then, } \sin x - \frac{1}{2} = 0 \quad \text{or} \quad \sin x - 1 = 0$$

$$\text{So, an equation is: } \left(\sin x - \frac{1}{2}\right)(\sin x - 1) = 0$$

This can be written as: $(2 \sin x - 1)(\sin x - 1) = 0$ or $2 \sin^2 x - 3 \sin x + 1 = 0$

17. Determine the number of roots each equation has over the domain $0 \leq x < 2\pi$, where $1 < a < b$.

a) $(a \cos x - b)(b \sin x + a) = 0$ b) $(b \sin^2 x - 1)(a \tan x + b) = 0$

Solve the equation.

$$\text{Either } a \cos x - b = 0$$

$$\cos x = \frac{b}{a}$$

But $b > a$, so $\frac{b}{a} > 1$, and there is no real solution

$$\text{Or } b \sin x + a = 0$$

$$\sin x = -\frac{a}{b}$$

Since $b > a$, there is a root in the two quadrants where the sine ratio is negative.

So, there are 2 roots.

Solve the equation.

$$\text{Either } b \sin^2 x - 1 = 0$$

$$\sin x = \pm \sqrt{\frac{1}{b}}$$

Since $b > 1$, $0 < \sqrt{\frac{1}{b}} < 1$, and there is a root in each quadrant.

$$\text{Or } a \tan x + b = 0$$

$$\tan x = -\frac{b}{a}$$

There is a root in the two quadrants where the tangent ratio is negative.

So, there is a total of 6 roots.