

Lesson 7.3 Exercises, pages 611–618

A

3. Write each expression in terms of a single trigonometric function.

a) $\frac{\cos \theta}{\sin \theta}$

$= \cot \theta$

b) $\frac{\sin^2 \theta}{\cos^2 \theta}$

$= \tan^2 \theta$

$$\begin{aligned} \text{c) } \sin^2\theta \sec \theta \cos \theta \csc \theta &= (\sin^2\theta)\left(\frac{1}{\cos \theta}\right)(\cos \theta)\left(\frac{1}{\sin \theta}\right) = \sin \theta \\ \text{d) } \frac{\sin^2\theta}{\tan^2\theta} &= \frac{\sin^2\theta}{\frac{\sin^2\theta}{\cos^2\theta}} = \cos^2\theta \end{aligned}$$

4. Determine the non-permissible values of θ .

a) $\sec \theta$

$$\begin{aligned} \sec \theta &= \frac{1}{\cos \theta}, \text{ so} \\ \text{non-permissible values} &\text{ occur when } \cos \theta = 0, \\ \text{so } \theta &= \frac{\pi}{2} + \pi k, k \in \mathbb{Z} \end{aligned}$$

b) $\tan \theta$

$$\begin{aligned} \tan \theta &= \frac{\sin \theta}{\cos \theta}, \text{ so non-permissible} \\ \text{values occur when } \cos \theta &= 0, \\ \text{so } \theta &= \frac{\pi}{2} + \pi k, k \in \mathbb{Z} \end{aligned}$$

c) $\frac{\csc \theta}{\cos \theta}$

$$\begin{aligned} \csc \theta &= \frac{1}{\sin \theta}, \text{ so} \\ \text{non-permissible values occur} &\text{ when } \sin \theta = 0, \text{ so } \theta = \pi k, \\ k \in \mathbb{Z}; \text{ or when } \cos \theta &= 0, \\ \text{so } \theta &= \frac{\pi}{2} + \pi k, k \in \mathbb{Z} \end{aligned}$$

d) $\frac{\sec \theta}{\sin \theta}$

$$\begin{aligned} \sec \theta &= \frac{1}{\cos \theta}, \text{ so non-permissible} \\ \text{values occur when } \cos \theta &= 0, \text{ so} \\ \theta &= \frac{\pi}{2} + \pi k, k \in \mathbb{Z}; \text{ or when} \\ \sin \theta &= 0, \text{ so } \theta = \pi k, k \in \mathbb{Z} \end{aligned}$$

5. Verify each identity for the given value of θ .

a) $\tan \theta \csc \theta \sec \theta = \sec^2\theta; \theta = 150^\circ$

Substitute: $\theta = 150^\circ$

$$\begin{aligned} \text{L.S.} &= \tan \theta \csc \theta \sec \theta \\ &= (\tan 150^\circ)(\csc 150^\circ)(\sec 150^\circ) \\ &= (-\tan 30^\circ)\left(\frac{1}{\sin 150^\circ}\right)\left(\frac{1}{\cos 150^\circ}\right) \\ &= (-\tan 30^\circ)\left(\frac{1}{\sin 30^\circ}\right)\left(\frac{-1}{\cos 30^\circ}\right) \\ &= \left(\frac{-1}{\sqrt{3}}\right)(2)\left(\frac{-2}{\sqrt{3}}\right) \\ &= \frac{4}{3} \\ \text{R.S.} &= \sec^2\theta \\ &= \sec^2(150^\circ) \\ &= \frac{1}{\cos^2(150^\circ)} \\ &= \left(\frac{-1}{\cos 30^\circ}\right)^2 \\ &= \left(\frac{-2}{\sqrt{3}}\right)^2 \\ &= \frac{4}{3} \end{aligned}$$

The left side is equal to the right side, so $\theta = 150^\circ$ is verified.

$$\text{b) } \frac{\tan \theta \csc^2 \theta}{\sec^2 \theta} = \cot \theta; \theta = \frac{4\pi}{3}$$

$$\text{Substitute: } \theta = \frac{4\pi}{3}$$

$$\begin{aligned} \text{L.S.} &= \frac{\tan \theta \csc^2 \theta}{\sec^2 \theta} \\ &= \frac{\left(\tan \frac{4\pi}{3}\right)\left(\csc^2\left(\frac{4\pi}{3}\right)\right)}{\sec^2\left(\frac{4\pi}{3}\right)} \\ &= \frac{\left(\tan \frac{\pi}{3}\right)\left(-\cos \frac{\pi}{3}\right)^2}{\left(-\sin \frac{\pi}{3}\right)^2} \\ &= \frac{(\sqrt{3})\left(-\frac{1}{2}\right)^2}{\left(-\frac{\sqrt{3}}{2}\right)^2} \\ &= \frac{1}{\sqrt{3}} \end{aligned}$$

$$\begin{aligned} \text{R.S.} &= \cot \theta \\ &= \cot \frac{4\pi}{3} \\ &= \frac{1}{\tan \frac{\pi}{3}} \\ &= \frac{1}{\sqrt{3}} \end{aligned}$$

The left side is equal to the right side, so $\theta = \frac{4\pi}{3}$ is verified.

B

6. Prove each identity in question 5.

$$\begin{aligned} \text{a) } \tan \theta \csc \theta \sec \theta &= \sec^2 \theta \\ \text{L.S.} &= \tan \theta \csc \theta \sec \theta \\ &= \left(\frac{\sin \theta}{\cos \theta}\right)\left(\frac{1}{\sin \theta}\right)\left(\frac{1}{\cos \theta}\right) \\ &= \frac{1}{\cos^2 \theta} \\ &= \sec^2 \theta \\ &= \text{R.S.} \end{aligned}$$

The left side is equal to the right side, so the identity is proved.

$$\begin{aligned} \text{b) } \frac{\tan \theta \csc^2 \theta}{\sec^2 \theta} &= \cot \theta \\ \text{L.S.} &= \frac{\tan \theta \csc^2 \theta}{\sec^2 \theta} \\ &= \frac{\left(\frac{\sin \theta}{\cos \theta}\right)\left(\frac{1}{\sin \theta}\right)^2}{\left(\frac{1}{\cos \theta}\right)^2} \\ &= \frac{1}{(\sin \theta)(\cos \theta)} \\ &= \frac{1}{\left(\frac{1}{\cos \theta}\right)^2} \\ &= \frac{\cos \theta}{\sin \theta} \\ &= \cot \theta \\ &= \text{R.S.} \end{aligned}$$

The left side is equal to the right side, so the identity is proved.

7. For each identity:

i) Verify the identity using graphing technology.

ii) Prove the identity.

a) $1 - \sin \theta = (\sin \theta)(\csc \theta - 1)$ b) $-\cot \theta = \frac{1 - \cot \theta}{1 - \tan \theta}$

i) Graph: $y = 1 - \sin \theta$ and

$$y = \sin \theta \left(\frac{1}{\sin \theta} - 1 \right)$$

The graphs coincide, so the identity is verified.

ii) R.S. = $(\sin \theta)(\csc \theta - 1)$

$$= (\sin \theta) \left(\frac{1}{\sin \theta} - 1 \right)$$

$$= 1 - \sin \theta$$

$$= \text{L.S.}$$

The left side is equal to the right side, so the identity is proved.

i) Graph: $y = \frac{-1}{\tan \theta}$ and $y = \frac{1 - \frac{1}{\tan \theta}}{1 - \tan \theta}$

The graphs coincide, so the identity is verified.

ii) R.S. = $\frac{1 - \cot \theta}{1 - \tan \theta}$

$$= \frac{1 - \frac{1}{\tan \theta}}{1 - \tan \theta}$$

$$= \frac{\tan \theta - 1}{(\tan \theta)(1 - \tan \theta)}$$

$$= \frac{-1}{\tan \theta}$$

$$= -\cot \theta$$

$$= \text{L.S.}$$

The left side is equal to the right side, so the identity is proved.

8. For each identity:

i) Verify the identity for $\theta = 45^\circ$.

ii) Prove the identity.

a) $\frac{\cot \theta}{\cos \theta} - \csc \theta = 0$ b) $\tan^2 \theta \cos^2 \theta + \sin^2 \theta = \frac{2}{\csc^2 \theta}$

i) Substitute: $\theta = 45^\circ$

$$\text{L.S.} = \frac{\cot \theta}{\cos \theta} - \csc \theta$$

$$= \frac{\cot 45^\circ}{\cos 45^\circ} - \csc 45^\circ$$

$$= \frac{1}{\frac{1}{\sqrt{2}}} - \sqrt{2}$$

$$= \sqrt{2} - \sqrt{2}$$

$$= 0$$

$$= \text{R.S.}$$

The left side is equal to the right side, so $\theta = 45^\circ$ is verified.

i) Substitute: $\theta = 45^\circ$

$$\text{L.S.} = \tan^2 \theta \cos^2 \theta + \sin^2 \theta$$

$$= (\tan 45^\circ)^2 (\cos 45^\circ)^2 + (\sin 45^\circ)^2$$

$$= (1) \left(\frac{1}{2} \right) + \left(\frac{1}{2} \right)$$

$$= 1$$

$$\text{R.S.} = \frac{2}{\csc^2 \theta}$$

$$= \frac{2}{(\csc 45^\circ)^2}$$

$$= \frac{2}{(\sqrt{2})^2}$$

$$= \frac{2}{2}$$

$$= 1$$

The left side is equal to the right side, so $\theta = 45^\circ$ is verified.



$$\begin{aligned}
 \text{ii) L.S.} &= \frac{\cot \theta}{\cos \theta} - \csc \theta \\
 &= \frac{\cos \theta}{\sin \theta} - \frac{1}{\sin \theta} \\
 &= \frac{1}{\sin \theta} - \frac{1}{\sin \theta} \\
 &= 0 \\
 &= \text{R.S.}
 \end{aligned}$$

The left side is equal to the right side, so the identity is proved.

$$\begin{aligned}
 \text{ii) L.S.} &= \tan^2 \theta \cos^2 \theta + \sin^2 \theta \\
 &= \left(\frac{\sin^2 \theta}{\cos^2 \theta} \right) (\cos^2 \theta) + \sin^2 \theta \\
 &= \sin^2 \theta + \sin^2 \theta \\
 &= 2 \sin^2 \theta \\
 &= \frac{2}{\csc^2 \theta} \\
 &= \text{R.S.}
 \end{aligned}$$

The left side is equal to the right side, so the identity is proved.

9. For each identity:

i) Verify the identity for $\theta = \frac{7\pi}{6}$.

ii) Prove the identity.

a) $\csc \theta = \frac{\csc \theta - 1}{1 - \sin \theta}$

i) Substitute: $\theta = \frac{7\pi}{6}$

$$\begin{aligned}
 \text{L.S.} &= \csc \theta \\
 &= \csc \frac{7\pi}{6} \\
 &= -\csc \frac{\pi}{6} \\
 &= -2 \\
 \text{R.S.} &= \frac{\csc \theta - 1}{1 - \sin \theta} \\
 &= \frac{\csc \frac{7\pi}{6} - 1}{1 - \sin \frac{7\pi}{6}} \\
 &= \frac{-2 - 1}{1 - \left(-\frac{1}{2}\right)} \\
 &= -2
 \end{aligned}$$

The left side is equal to the right side, so $\theta = \frac{7\pi}{6}$ is verified.

ii) $\text{R.S.} = \frac{\csc \theta - 1}{1 - \sin \theta}$

$$\begin{aligned}
 &= \frac{1}{\sin \theta} - 1 \\
 &= \frac{1 - \sin \theta}{1 - \sin \theta} \\
 &= \frac{1 - \sin \theta}{(\sin \theta)(1 - \sin \theta)} \\
 &= \frac{1}{\sin \theta} \\
 &= \csc \theta \\
 &= \text{L.S.}
 \end{aligned}$$

The left side is equal to the right side, so the identity is proved.

b) $\frac{\cos \theta - \cot \theta}{1 - \sin \theta} = -\cot \theta$

i) Substitute: $\theta = \frac{7\pi}{6}$

$$\begin{aligned}
 \text{L.S.} &= \frac{\cos \theta - \cot \theta}{1 - \sin \theta} \\
 &= \frac{\cos \frac{7\pi}{6} - \cot \frac{7\pi}{6}}{1 - \sin \frac{7\pi}{6}} \\
 &= \frac{-\frac{\sqrt{3}}{2} - \sqrt{3}}{1 - \left(-\frac{1}{2}\right)} \\
 &= -\sqrt{3} \\
 \text{R.S.} &= -\cot \theta \\
 &= -\cot \frac{7\pi}{6} \\
 &= -\sqrt{3}
 \end{aligned}$$

The left side is equal to the right side, so $\theta = \frac{7\pi}{6}$ is verified.

ii) $\text{L.S.} = \frac{\cos \theta - \cot \theta}{1 - \sin \theta}$

$$\begin{aligned}
 &= \frac{\cos \theta - \frac{\cos \theta}{\sin \theta}}{1 - \sin \theta} \\
 &= \frac{(\cos \theta)(\sin \theta) - \cos \theta}{(1 - \sin \theta)(\sin \theta)} \\
 &= \frac{(\cos \theta)(\sin \theta - 1)}{(1 - \sin \theta)(\sin \theta)} \\
 &= \frac{-\cos \theta}{\sin \theta} \\
 &= -\cot \theta \\
 &= \text{R.S.}
 \end{aligned}$$

The left side is equal to the right side, so the identity is proved.

10. Use algebra to solve each equation over the domain $0 \leq x < 2\pi$.
Give the roots to the nearest hundredth where necessary.

a) $\tan x = \cot x$

Assume $\tan x \neq 0$, then divide by $\tan x$.

$$\frac{\tan x}{\tan x} = \frac{\cot x}{\tan x}$$

$$1 = \frac{1}{\tan^2 x}$$

$$\tan^2 x = 1$$

$$\tan x = \pm 1$$

$$x = \frac{\pi}{4}, x = \frac{3\pi}{4}, x = \frac{5\pi}{4}, \text{ or}$$

$$x = \frac{7\pi}{4}$$

For $\tan x = 0$, $x = 0$ or $x = \pi$

Verify by substitution that neither value of x is a root of the given equation.

Verify the roots by substitution.

b) $\cos x + \sqrt{3} \sin x = 0$

Assume $\cos x \neq 0$, then divide by $\cos x$.

$$\frac{\cos x}{\cos x} + \sqrt{3} \frac{\sin x}{\cos x} = 0$$

$$1 + \sqrt{3} \tan x = 0$$

$$\tan x = -\frac{1}{\sqrt{3}}$$

$$x = \frac{5\pi}{6} \text{ or } x = \frac{11\pi}{6}$$

$$\text{For } \cos x = 0, x = \frac{\pi}{2} \text{ or } x = \frac{3\pi}{2}$$

Verify by substitution that neither value of x is a root of the given equation.

Verify the roots by substitution.

c) $2 \cos x = 7 - 3 \sec x$

$$2 \cos x = 7 - \frac{3}{\cos x}$$

Multiply by $\cos x$, then collect terms on one side.

$$2 \cos^2 x - 7 \cos x + 3 = 0$$

$$(2 \cos x - 1)(\cos x - 3) = 0$$

$$\text{Either } 2 \cos x - 1 = 0$$

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3} \text{ or } x = \frac{5\pi}{3}$$

$$\text{Or } \cos x - 3 = 0$$

$$\cos x = 3$$

This equation has no solution.

Verify the roots by substitution.

d) $\sin^2 x = \sin x \cos x$

$$\sin^2 x - \sin x \cos x = 0$$

$$(\sin x)(\sin x - \cos x) = 0$$

$$\text{Either } \sin x = 0$$

$$x = 0 \text{ or } x = \pi$$

$$\text{Or } \sin x - \cos x = 0$$

$$\sin x = \cos x$$

Assume $\cos x \neq 0$, then divide by $\cos x$.

$$\tan x = 1$$

$$x = \frac{\pi}{4} \text{ or } x = \frac{5\pi}{4}$$

$$\text{For } \cos x = 0, x = \frac{\pi}{2} \text{ or } x = \frac{3\pi}{2}$$

Verify by substitution that neither value of x is a root of the given equation.

Verify the roots by substitution.

11. Identify any errors in this proof, then write a correct algebraic proof.

To prove: $\frac{\sin \theta}{1 - \sin \theta} = \frac{1}{\csc \theta - 1}$

$$\begin{aligned} \text{L.S.} &= \frac{\sin \theta}{1 - \sin \theta} \\ &= \frac{\sin \theta}{1} - \frac{\sin \theta}{\sin \theta} \\ &= \sin \theta - 1 \\ &= \frac{1}{\csc \theta} - 1 \\ &= \frac{1}{\csc \theta - 1} \\ &= \text{R.S.} \end{aligned}$$

Correct proof:

$$\begin{aligned} \text{R.S.} &= \frac{1}{\csc \theta - 1} \\ &= \frac{1}{\frac{1}{\sin \theta} - 1} \\ &= \frac{1}{\frac{1 - \sin \theta}{\sin \theta}} \\ &= \frac{\sin \theta}{1 - \sin \theta} \\ &= \text{L.S.} \end{aligned}$$

From the 1st line of the proof to the 2nd line, $\frac{\sin \theta}{1 - \sin \theta}$ cannot be written as $\frac{\sin \theta}{1} - \frac{\sin \theta}{\sin \theta}$.

From the 4th line of the proof to the 5th line, $\frac{1}{\csc \theta} - 1$ cannot be written as $\frac{1}{\csc \theta - 1}$.

12. Identify which equation below is an identity. Justify your answer.

Prove the identity. Solve the other equation over the domain

$-\pi \leq x \leq \pi$. Give the roots to the nearest hundredth.

a) $\frac{2 \sin^2 x + 1}{\sin x} = 2 \csc^2 x - 1$ b) $\frac{\sin^2 x + 1}{\sin x} = \frac{1 + \csc^2 x}{\csc x}$

I graphed each side of the equation. The graphs do not coincide, so this is an equation.

From the graphs, the roots are approximately: $x = 0.81$ and $x = 2.33$

I graphed each side of the equation. The graphs appear to coincide, so this is probably the identity.

$$\begin{aligned} \text{R.S.} &= \frac{1 + \csc^2 x}{\csc x} \\ &= \frac{1 + \frac{1}{\sin^2 x}}{\frac{1}{\sin x}} \end{aligned}$$

Multiply numerator and denominator by $\sin^2 x$.

$$\begin{aligned} \text{R.S.} &= \frac{\sin^2 x + 1}{\sin x} \\ &= \text{L.S.} \end{aligned}$$

The left side is equal to the right side, so the identity is proved.

C

13. Here are two identities that involve the cotangent ratio:

$$\cot \theta = \frac{1}{\tan \theta} \quad \text{and} \quad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

a) Show how you can derive one identity from the other.

$$\cot \theta = \frac{1}{\tan \theta} \quad \text{Write } \tan \theta = \frac{\sin \theta}{\cos \theta}.$$

$$\cot \theta = \frac{1}{\frac{\sin \theta}{\cos \theta}} \quad \text{Multiply numerator and denominator by } \cos \theta.$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

- b) Determine the non-permissible values of θ for each identity.
Explain why these values are different. How could you illustrate this using graphing technology?

$$\text{For } \cot \theta = \frac{1}{\tan \theta}$$

$$\text{Since } \tan \theta = \frac{\sin \theta}{\cos \theta}, \text{ then}$$

$$\sin \theta \neq 0 \text{ and } \cos \theta \neq 0$$

$$\text{So, } \theta \neq \pi k, k \in \mathbb{Z} \text{ and } \theta \neq \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$$

$$\text{For } \cot \theta = \frac{\cos \theta}{\sin \theta}$$

$$\sin \theta \neq 0$$

$$\text{So, } \theta \neq \pi k, k \in \mathbb{Z}$$

The values are different because there are two restrictions for $\frac{1}{\tan \theta}$ and only one restriction for $\frac{\cos \theta}{\sin \theta}$.

When I graph $y = \frac{1}{\tan \theta}$, and set the TABLE for intervals of $\frac{\pi}{2}$, it shows

ERROR for all values of X in the table. When I graph $y = \frac{\cos \theta}{\sin \theta}$, with the same TABLE settings, it shows ERROR only for $X = \pi k, k \in \mathbb{Z}$.