

Lesson 7.5 Exercises, pages 641–649

A

3. Verify each identity for the given values of α and β .

a) $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$;
for $\alpha = 90^\circ$ and $\beta = 30^\circ$

Substitute: $\alpha = 90^\circ$ and $\beta = 30^\circ$

$$\begin{array}{l} \text{L.S.} = \sin(\alpha + \beta) \\ = \sin(90^\circ + 30^\circ) \\ = \sin 120^\circ \\ = \frac{\sqrt{3}}{2} \end{array} \qquad \begin{array}{l} \text{R.S.} = \sin \alpha \cos \beta + \cos \alpha \sin \beta \\ = \sin 90^\circ \cos 30^\circ + \cos 90^\circ \sin 30^\circ \\ = (1)\left(\frac{\sqrt{3}}{2}\right) + (0)\left(\frac{1}{2}\right) \\ = \frac{\sqrt{3}}{2} + 0 \\ = \frac{\sqrt{3}}{2} \end{array}$$

The left side is equal to the right side, so the identity is verified.

b) $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$; for $\alpha = \frac{\pi}{2}$ and $\beta = \frac{\pi}{4}$

Substitute: $\alpha = \frac{\pi}{2}$ and $\beta = \frac{\pi}{4}$

$$\begin{array}{l} \text{L.S.} = \cos(\alpha - \beta) \\ = \cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right) \\ = \sin \frac{\pi}{4} \\ = \frac{1}{\sqrt{2}} \end{array} \qquad \begin{array}{l} \text{R.S.} = \cos \alpha \cos \beta + \sin \alpha \sin \beta \\ = \cos \frac{\pi}{2} \cos \frac{\pi}{4} + \sin \frac{\pi}{2} \sin \frac{\pi}{4} \\ = (0)\left(\frac{1}{\sqrt{2}}\right) + (1)\left(\frac{1}{\sqrt{2}}\right) \\ = \frac{1}{\sqrt{2}} \end{array}$$

The left side is equal to the right side, so the identity is verified.

c) $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$; for $\alpha = \frac{2\pi}{3}$ and $\beta = \frac{\pi}{3}$

Substitute: $\alpha = \frac{2\pi}{3}$ and $\beta = \frac{\pi}{3}$

$$\begin{array}{l} \text{L.S.} = \tan(\alpha + \beta) \\ = \tan\left(\frac{2\pi}{3} + \frac{\pi}{3}\right) \\ = \tan \pi \\ = 0 \end{array} \qquad \begin{array}{l} \text{R.S.} = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \\ = \frac{\tan \frac{2\pi}{3} + \tan \frac{\pi}{3}}{1 - \tan \frac{2\pi}{3} \tan \frac{\pi}{3}} \\ = \frac{-\sqrt{3} + \sqrt{3}}{1 - (-\sqrt{3})(\sqrt{3})} \\ = 0 \end{array}$$

The left side is equal to the right side, so the identity is verified.

4. Simplify each expression.

$$\begin{aligned} \text{a) } \cos 8\theta \cos 2\theta + \sin 8\theta \sin 2\theta &= \cos(8\theta - 2\theta) \\ &= \cos 6\theta \end{aligned} \qquad \begin{aligned} \text{b) } \cos \theta \sin 4\theta - \sin \theta \cos 4\theta &= \sin 4\theta \cos \theta - \cos 4\theta \sin \theta \\ &= \sin(4\theta - \theta) \\ &= \sin 3\theta \end{aligned}$$

$$\begin{aligned} \text{c) } \frac{\tan 7x - \tan 3x}{1 + \tan 7x \tan 3x} &= \tan(7x - 3x) \\ &= \tan 4x \end{aligned} \qquad \begin{aligned} \text{d) } \sin 5x \sin 3x - \cos 5x \cos 3x &= -(\cos 5x \cos 3x - \sin 5x \sin 3x) \\ &= -\cos(5x + 3x) \\ &= -\cos 8x \end{aligned}$$

B

5. Simplify each expression, then determine its exact value.

$$\begin{aligned} \text{a) } \cos 75^\circ \cos 15^\circ - \sin 75^\circ \sin 15^\circ &= \cos(75^\circ + 15^\circ) \\ &= \cos 90^\circ \\ &= 0 \end{aligned} \qquad \begin{aligned} \text{b) } \sin \pi \cos \frac{\pi}{4} + \cos \pi \sin \frac{\pi}{4} &= \sin\left(\pi + \frac{\pi}{4}\right) \\ &= \sin \frac{5\pi}{4} \\ &= -\frac{1}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} \text{c) } \cos \frac{\pi}{3} \sin \frac{\pi}{6} - \sin \frac{\pi}{3} \cos \frac{\pi}{6} &= -\left(\sin \frac{\pi}{3} \cos \frac{\pi}{6} - \cos \frac{\pi}{3} \sin \frac{\pi}{6}\right) \\ &= -\sin\left(\frac{\pi}{3} - \frac{\pi}{6}\right) \\ &= -\sin \frac{\pi}{6} \\ &= -\frac{1}{2} \end{aligned} \qquad \begin{aligned} \text{d) } \frac{\tan \pi + \tan \frac{\pi}{3}}{1 - \tan \pi \tan \frac{\pi}{3}} &= \tan\left(\pi + \frac{\pi}{3}\right) \\ &= \tan \frac{4\pi}{3} \\ &= \sqrt{3} \end{aligned}$$

6. a) Expand $\sin\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$ to verify that $\sin \pi = 0$.

$$\begin{aligned} \sin\left(\frac{\pi}{2} + \frac{\pi}{2}\right) &= \sin \frac{\pi}{2} \cos \frac{\pi}{2} + \cos \frac{\pi}{2} \sin \frac{\pi}{2} \\ \sin \pi &= (1)(0) + (0)(1) \\ &= 0 \end{aligned}$$

b) Expand $\cos\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$ to verify that $\cos \pi = -1$.

$$\begin{aligned} \cos\left(\frac{\pi}{2} + \frac{\pi}{2}\right) &= \cos \frac{\pi}{2} \cos \frac{\pi}{2} - \sin \frac{\pi}{2} \sin \frac{\pi}{2} \\ \cos \pi &= (0)(0) - (1)(1) \\ &= -1 \end{aligned}$$

c) Expand $\cos\left(\frac{\pi}{2} - \frac{\pi}{3}\right)$ to verify that $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$.

$$\begin{aligned}\cos\left(\frac{\pi}{2} - \frac{\pi}{3}\right) &= \cos\frac{\pi}{2}\cos\frac{\pi}{3} + \sin\frac{\pi}{2}\sin\frac{\pi}{3} \\ \cos\frac{\pi}{6} &= (0)\left(\frac{1}{2}\right) + (1)\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

d) Expand $\tan\left(\frac{\pi}{6} + \frac{\pi}{6}\right)$ to verify that $\tan\frac{\pi}{3} = \sqrt{3}$.

$$\begin{aligned}\tan\left(\frac{\pi}{6} + \frac{\pi}{6}\right) &= \frac{\tan\frac{\pi}{6} + \tan\frac{\pi}{6}}{1 - \tan\frac{\pi}{6}\tan\frac{\pi}{6}} \\ \tan\frac{\pi}{3} &= \frac{\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}}{1 - \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)} \\ &= \frac{\frac{2}{\sqrt{3}}}{\frac{2}{3}}, \text{ or } \sqrt{3}\end{aligned}$$

7. Determine each exact value.

a) $\cos 75^\circ$

$$\begin{aligned}&= \cos(45^\circ + 30^\circ) \\ &= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ \\ &= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) \\ &= \frac{\sqrt{3} - 1}{2\sqrt{2}}\end{aligned}$$

b) $\sin 15^\circ$

$$\begin{aligned}&= \sin(45^\circ - 30^\circ) \\ &= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ \\ &= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) \\ &= \frac{\sqrt{3} - 1}{2\sqrt{2}}\end{aligned}$$

c) $\tan\frac{\pi}{12}$

$$\begin{aligned}&= \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\ &= \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1 + \tan\frac{\pi}{3}\tan\frac{\pi}{4}} \\ &= \frac{\sqrt{3} - 1}{1 + (\sqrt{3})(1)} \\ &= \frac{\sqrt{3} - 1}{\sqrt{3} + 1}\end{aligned}$$

d) $\tan\frac{5\pi}{12}$

$$\begin{aligned}&= \tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right) \\ &= \frac{\tan\frac{\pi}{6} + \tan\frac{\pi}{4}}{1 - \tan\frac{\pi}{6}\tan\frac{\pi}{4}} \\ &= \frac{\frac{1}{\sqrt{3}} + 1}{1 - \left(\frac{1}{\sqrt{3}}\right)(1)} \\ &= \frac{1 + \sqrt{3}}{\sqrt{3} - 1}\end{aligned}$$

9. Prove each identity.

a) $\cos \theta = \sin \left(\frac{\pi}{2} + \theta \right)$

Use: $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$
 Substitute: $\alpha = \frac{\pi}{2}, \beta = \theta$
 R.S. = $\sin \frac{\pi}{2} \cos \theta + \cos \frac{\pi}{2} \sin \theta$
 = $(1) \cos \theta + (0) \sin \theta$
 = $\cos \theta$
 = L.S.

The left side is equal to the right side, so the identity is proved.

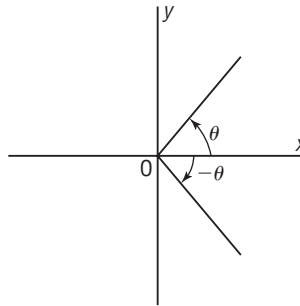
b) $-\tan \theta = \tan(\pi - \theta)$

Use: $\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$
 Substitute: $\alpha = \pi, \beta = \theta$
 R.S. = $\frac{\tan \pi - \tan \theta}{1 + \tan \pi \tan \theta}$
 = $\frac{0 - \tan \theta}{1 + (0) \tan \theta}$
 = $-\tan \theta$
 = L.S.

The left side is equal to the right side, so the identity is proved.

10. For each identity below:

- i) Use the diagram at the right to explain why each identity is true.
- ii) Prove the identity.



a) $\cos(-\theta) = \cos \theta$

- i) The cosine of an angle whose terminal arm lies in Quadrant 4 is positive and equal to the cosine of its reference angle in Quadrant 1.

ii) L.S. = $\cos(-\theta)$
 = $\cos(0 - \theta)$
 = $\cos 0 \cos \theta + \sin 0 \sin \theta$
 = $(1) \cos \theta + (0) \sin \theta$
 = $\cos \theta$
 = R.S.

The left side is equal to the right side, so the identity is proved.

b) $\sin(-\theta) = -\sin \theta$

- i) The sine of an angle whose terminal arm lies in Quadrant 4 is negative and has the same numerical value as the sine of its reference angle in Quadrant 1.

ii) L.S. = $\sin(-\theta)$
 = $\sin(0 - \theta)$
 = $\sin 0 \cos \theta - \cos 0 \sin \theta$
 = $(0) \cos \theta - (1) \sin \theta$
 = $-\sin \theta$
 = R.S.

The left side is equal to the right side, so the identity is proved.

c) $\tan(-\theta) = -\tan \theta$

i) The tangent of an angle whose terminal arm lies in Quadrant 4 is negative and has the same numerical value as the tangent of its reference angle in Quadrant 1.

$$\begin{aligned} \text{ii) L.S.} &= \tan(-\theta) \\ &= \frac{\sin(-\theta)}{\cos(-\theta)} \\ &= \frac{-\sin \theta}{\cos \theta} \\ &= -\tan \theta \\ &= \text{R.S.} \end{aligned}$$

The left side is equal to the right side, so the identity is proved.

- 11.** If $f(x)$ is an even function, then $f(-x) = f(x)$
 If $f(x)$ is an odd function, then $f(-x) = -f(x)$
 Use the identities in question 10 to classify each of the sine, cosine, and tangent functions as odd or even.

Since $\sin(-x) = -\sin x$, the sine function is an odd function

Since $\cos(-x) = \cos x$, the cosine function is an even function

Since $\tan(-x) = -\tan x$, the tangent function is an odd function

- 12.** Solve this equation over the domain $-90^\circ < x < 270^\circ$:

$$\frac{\tan 4x - \tan 3x}{1 + \tan 4x \tan 3x} = \sqrt{3}$$

$$\begin{aligned} \tan(4x - 3x) &= \sqrt{3} \\ \tan x &= \sqrt{3} \\ x &= 60^\circ \text{ or } x = 240^\circ \end{aligned}$$

- 13.** Given $\tan \alpha = \frac{4}{3}$ and $\tan \beta = -\frac{5}{12}$, where angle α is in standard position in Quadrant 1 and angle β is in standard position with its terminal arm in Quadrant 2, determine the exact value of $\cos(\alpha - \beta)$.

Use: $x^2 + y^2 = r^2$

For angle α , substitute:

$$\begin{aligned} x &= 3, y = 4 \\ 3^2 + 4^2 &= r^2 \\ r &= 5 \end{aligned}$$

So, $\sin \alpha = \frac{4}{5}$ and $\cos \alpha = \frac{3}{5}$

For angle β , substitute:

$$\begin{aligned} x &= -12, y = 5 \\ (-12)^2 + 5^2 &= r^2 \\ r &= 13 \end{aligned}$$

So, $\sin \beta = \frac{5}{13}$ and $\cos \beta = -\frac{12}{13}$

Substitute for α and β in:

$$\begin{aligned} \cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \\ &= \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right) + \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) \\ &= \frac{-36}{65} + \frac{20}{65} \\ &= -\frac{16}{65} \end{aligned}$$

14. Prove each identity.

a) $\sin\left(\frac{\pi}{4} + \theta\right) + \sin\left(\frac{\pi}{4} - \theta\right) = \sqrt{2} \cos \theta$

$$\begin{aligned} \text{L.S.} &= \sin \frac{\pi}{4} \cos \theta + \cos \frac{\pi}{4} \sin \theta + \sin \frac{\pi}{4} \cos \theta - \cos \frac{\pi}{4} \sin \theta \\ &= 2 \sin \frac{\pi}{4} \cos \theta \\ &= 2 \left(\frac{1}{\sqrt{2}}\right) \cos \theta \\ &= \sqrt{2} \cos \theta \\ &= \text{R.S.} \end{aligned}$$

The left side is equal to the right side, so the identity is proved.

b) $\cos\left(\frac{\pi}{6} + \theta\right) - \cos\left(\frac{\pi}{6} - \theta\right) = -\sin \theta$

$$\begin{aligned} \text{L.S.} &= \cos \frac{\pi}{6} \cos \theta - \sin \frac{\pi}{6} \sin \theta - \left(\cos \frac{\pi}{6} \cos \theta + \sin \frac{\pi}{6} \sin \theta\right) \\ &= -2 \sin \frac{\pi}{6} \sin \theta \\ &= -2 \left(\frac{1}{2}\right) \sin \theta \\ &= -\sin \theta \\ &= \text{R.S.} \end{aligned}$$

The left side is equal to the right side, so the identity is proved.

c) $\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$

$$\begin{aligned} \text{L.S.} &= \tan(\alpha - \beta) \\ &= \frac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)} && \text{Expand.} \\ &= \frac{\sin \alpha \cos \beta - \cos \alpha \sin \beta}{\cos \alpha \cos \beta + \sin \alpha \sin \beta} && \text{Divide numerator and denominator} \\ & && \text{by } \cos \alpha \cos \beta. \\ &= \frac{\frac{\sin \alpha \cos \beta}{\cos \alpha \cos \beta} - \frac{\cos \alpha \sin \beta}{\cos \alpha \cos \beta}}{\frac{\cos \alpha \cos \beta}{\cos \alpha \cos \beta} + \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}} && \text{Simplify.} \\ &= \frac{\frac{\sin \alpha}{\cos \alpha} - \frac{\sin \beta}{\cos \beta}}{1 + \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}} && \text{Use the tangent identity.} \\ &= \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} \\ &= \text{R.S.} \end{aligned}$$

The left side is equal to the right side, so the identity is proved.

15. Solve each equation over the domain $-\frac{3\pi}{2} \leq x < \frac{\pi}{2}$.

a) $\sin 7x \cos 5x - \cos 7x \sin 5x = -1$

$$\begin{aligned}\sin(7x - 5x) &= -1 \\ \sin 2x &= -1\end{aligned}$$

Consider $-3\pi \leq 2x < \pi$.

$$2x = -\frac{\pi}{2} \quad \text{or} \quad 2x = -\frac{5\pi}{2}$$

$$x = -\frac{\pi}{4} \quad \quad \quad x = -\frac{5\pi}{4}$$

The roots are: $x = -\frac{\pi}{4}$ and $x = -\frac{5\pi}{4}$

b) $\cos 2x \cos x - \sin 2x \sin x = 0$

$$\begin{aligned}\cos(2x + x) &= 0 \\ \cos 3x &= 0\end{aligned}$$

Consider $-\frac{9\pi}{2} \leq 3x < \frac{3\pi}{2}$.

$$3x = \frac{\pi}{2} \text{ or } -\frac{\pi}{2} \text{ or } -\frac{3\pi}{2} \text{ or } -\frac{5\pi}{2} \text{ or } -\frac{7\pi}{2} \text{ or } -\frac{9\pi}{2}$$

$$x = \frac{\pi}{6} \text{ or } -\frac{\pi}{6} \text{ or } -\frac{\pi}{2} \text{ or } -\frac{5\pi}{6} \text{ or } -\frac{7\pi}{6} \text{ or } -\frac{3\pi}{2}$$

The roots are: $x = \pm\frac{\pi}{6}$, $x = -\frac{\pi}{2}$, $x = -\frac{5\pi}{6}$, $x = -\frac{7\pi}{6}$, $x = -\frac{3\pi}{2}$

16. Here are two solutions for solving the equation $\sin(\pi + x) = \frac{1}{\sqrt{2}}$

over the set of real numbers. Are both solutions correct? Explain.

Solution 1

$$\sin(\pi + x) = \frac{1}{\sqrt{2}}$$

$$\text{Either } \pi + x = \frac{\pi}{4} + 2\pi k, k \in \mathbb{Z}$$

$$x = -\frac{3\pi}{4} + 2\pi k, k \in \mathbb{Z}$$

$$\text{Or } \pi + x = \frac{3\pi}{4} + 2\pi k, k \in \mathbb{Z}$$

$$x = -\frac{\pi}{4} + 2\pi k, k \in \mathbb{Z}$$

Solution 2

$$\sin(\pi + x) = \frac{1}{\sqrt{2}}$$

$$\sin \pi \cos x + \cos \pi \sin x = \frac{1}{\sqrt{2}}$$

$$(0) \cos x + (-1) \sin x = \frac{1}{\sqrt{2}}$$

$$-\sin x = \frac{1}{\sqrt{2}}$$

$$\sin x = -\frac{1}{\sqrt{2}}$$

$$x = \frac{5\pi}{4} + 2\pi k, k \in \mathbb{Z}$$

$$\text{or } x = \frac{7\pi}{4} + 2\pi k, k \in \mathbb{Z}$$

Both solutions are correct. For $k = 1$, $-\frac{3\pi}{4} + 2\pi k$, $k \in \mathbb{Z}$ becomes

$$x = \frac{5\pi}{4}, \text{ and } -\frac{\pi}{4} + 2\pi k, k \in \mathbb{Z} \text{ becomes } \frac{7\pi}{4}.$$

17. Determine the general solution of each equation over the set of real numbers.

a) $\cos(\pi + x) = \frac{1}{2}$

$$\cos \pi \cos x - \sin \pi \sin x = \frac{1}{2}$$

$$(-1) \cos x - (0) \sin x = \frac{1}{2}$$

$$\cos x = -\frac{1}{2}$$

$$x = \frac{2\pi}{3} + 2\pi k, k \in \mathbb{Z} \text{ or}$$

$$x = \frac{4\pi}{3} + 2\pi k, k \in \mathbb{Z}$$

b) $\tan(\pi + x) = -1$

$$\frac{\tan \pi + \tan x}{1 - \tan \pi \tan x} = -1$$

$$\frac{0 + \tan x}{1 - (0) \tan x} = -1$$

$$\tan x = -1$$

$$x = \frac{3\pi}{4} + \pi k, k \in \mathbb{Z}$$

C

18. Use algebra to determine the amplitude and the period of the graph of $y = \sin\left(\frac{\pi}{4} + x\right) + \sin\left(\frac{\pi}{4} - x\right)$. Describe your strategy.

Use graphing technology to check.

$$y = \sin\left(\frac{\pi}{4} + x\right) + \sin\left(\frac{\pi}{4} - x\right)$$

Use the sum and difference formulas.

$$y = \sin \frac{\pi}{4} \cos x + \cos \frac{\pi}{4} \sin x + \sin \frac{\pi}{4} \cos x - \cos \frac{\pi}{4} \sin x$$

$$y = 2 \sin \frac{\pi}{4} \cos x$$

$$y = 2 \left(\frac{1}{\sqrt{2}}\right) \cos x$$

$$y = \sqrt{2} \cos x$$

The amplitude is $\sqrt{2}$ and the period is 2π .