

Lesson 7.6 Exercises, pages 658–665

A

4. Write each expression as a single trigonometric ratio.

a) $\sin(\theta + \theta)$

$$= \sin 2\theta$$

b) $\sin \theta + \sin \theta$

$$= 2 \sin \theta$$

c) $\sin \theta \sin \theta$

$$= \sin^2 \theta$$

d) $\cos^2 \theta \cos \theta$

$$= \cos^3 \theta$$

e) $3 \sin \theta \sin \theta$

$$= 3 \sin^2 \theta$$

f) $3 \sin \theta + \sin \theta$

$$= 4 \sin \theta$$

5. Determine the exact value of each expression.

a) $2 \sin 45^\circ \cos 45^\circ$

$$= \sin 2(45^\circ)$$

$$= \sin 90^\circ$$

$$= 1$$

b) $\cos^2\left(\frac{\pi}{6}\right) \sin^2\left(\frac{\pi}{6}\right)$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 \left(\frac{1}{2}\right)^2$$

$$= \frac{3}{16}$$

c) $2 \cos^2\left(\frac{\pi}{8}\right) - 1$

$$= \cos 2\left(\frac{\pi}{8}\right)$$

$$= \cos \frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}}$$

d) $\frac{2 \tan 210^\circ}{1 - \tan^2(210^\circ)}$

$$= \tan 2(210^\circ)$$

$$= \tan 420^\circ$$

$$= \tan 60^\circ$$

$$= \sqrt{3}$$

6. Simplify each expression.

a) $-\cos^2 \theta - \sin^2 \theta$

$$= -(\cos^2 \theta + \sin^2 \theta)$$

$$= -1$$

b) $1 - \sin^2 \theta$

$$= \cos^2 \theta$$

c) $2 \cos^2 \theta - 2$

$$= 2(\cos^2 \theta - 1)$$

$$= -2 \sin^2 \theta$$

d) $4 \cos^2 \theta - 2$

$$= 2(2 \cos^2 \theta - 1)$$

$$= 2 \cos 2\theta$$

B

7. Verify each identity for the given value of θ .

a) $\sin 2\theta = 2 \sin \theta \cos \theta; \theta = 30^\circ$ b) $\cos 2\theta = \cos^2\theta - \sin^2\theta; \theta = \frac{5\pi}{4}$

$$\begin{aligned} \text{L.S.} &= \sin 2\theta \\ &= \sin 60^\circ \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

$$\begin{aligned} \text{R.S.} &= 2 \sin \theta \cos \theta \\ &= 2 \sin 30^\circ \cos 30^\circ \\ &= 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

The left side is equal to the right side, so the identity is verified.

$$\begin{aligned} \text{L.S.} &= \cos 2\theta \\ &= \cos \frac{10\pi}{4} \\ &= \cos \frac{\pi}{2} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{R.S.} &= \cos^2\theta - \sin^2\theta \\ &= \left(\cos \frac{5\pi}{4}\right)^2 - \left(\sin \frac{5\pi}{4}\right)^2 \\ &= \left(-\frac{1}{\sqrt{2}}\right)^2 - \left(-\frac{1}{\sqrt{2}}\right)^2 \\ &= 0 \end{aligned}$$

The left side is equal to the right side, so the identity is verified.

8. For the identity $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2\theta}$, determine the non-permissible values of θ over the set of real numbers, then verify the identity for $\theta = \frac{2\pi}{3}$.

$$\begin{aligned} \cos 2\theta &\neq 0 \\ 2\theta &\neq \frac{\pi}{2} + \pi k, \\ k &\in \mathbb{Z} \\ \theta &\neq \frac{\pi}{4} + \frac{\pi}{2}k, \\ k &\in \mathbb{Z} \end{aligned}$$

$$\begin{aligned} \cos \theta &\neq 0 \\ \theta &\neq \frac{\pi}{2} + \pi k, \\ k &\in \mathbb{Z} \end{aligned}$$

$$\begin{aligned} \tan^2\theta &\neq 1 \\ \tan \theta &\neq \pm 1 \\ \theta &\neq \frac{\pi}{4}(2k + 1), k \in \mathbb{Z} \end{aligned}$$

To verify, substitute $\theta = \frac{2\pi}{3}$ in each side of the identity.

$$\begin{aligned} \text{L.S.} &= \tan 2\theta \\ &= \tan \frac{4\pi}{3} \\ &= \sqrt{3} \end{aligned}$$

$$\begin{aligned} \text{R.S.} &= \frac{2 \tan \theta}{1 - \tan^2\theta} \\ &= \frac{2 \tan \frac{2\pi}{3}}{1 - \tan^2\left(\frac{2\pi}{3}\right)} \\ &= \frac{2(-\sqrt{3})}{1 - (-\sqrt{3})^2} \\ &= \sqrt{3} \end{aligned}$$

The left side is equal to the right side, so the identity is verified.

9. Given angle θ is in standard position with its terminal arm in Quadrant 2 and $\tan \theta = -\frac{2}{5}$, determine the exact value of each trigonometric ratio.

- a) $\sin 2\theta$ b) $\cos 2\theta$ c) $\tan 2\theta$

Since the terminal arm of θ lies in Quadrant 2, $\sin \theta$ is positive. Use mental math and the Pythagorean Theorem to determine the value of r , which is $\sqrt{29}$.

$$\text{So, } \sin \theta = \frac{2}{\sqrt{29}} \text{ and } \cos \theta = -\frac{5}{\sqrt{29}}$$

- a) Substitute values of $\sin \theta$ and $\cos \theta$ in:
 $\sin 2\theta = 2 \sin \theta \cos \theta$
 $= 2\left(\frac{2}{\sqrt{29}}\right)\left(-\frac{5}{\sqrt{29}}\right)$
 $= -\frac{20}{29}$
- b) Substitute the value of $\cos \theta$ in:
 $\cos 2\theta = 2 \cos^2 \theta - 1$
 $= 2\left(-\frac{5}{\sqrt{29}}\right)^2 - 1$
 $= \frac{50}{29} - 1, \text{ or } \frac{21}{29}$

- c) Substitute $\tan \theta = -\frac{2}{5}$ in $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$:

$$\tan 2\theta = \frac{2\left(-\frac{2}{5}\right)}{1 - \left(-\frac{2}{5}\right)^2}$$

$$= \frac{-\frac{4}{5}}{\frac{21}{25}}, \text{ or } -\frac{20}{21}$$

10. Prove each identity.

a) $2 \cot \theta \sin^2 \theta = \sin 2\theta$

$$\begin{aligned} \text{L.S.} &= 2 \cot \theta \sin^2 \theta \\ &= 2 \left(\frac{\cos \theta}{\sin \theta}\right) \sin^2 \theta \\ &= 2 \cos \theta \sin \theta \\ &= \sin 2\theta \\ &= \text{R.S.} \end{aligned}$$

The left side is equal to the right side, so the identity is proved.

b) $1 + \sin 2\theta = (\sin \theta + \cos \theta)^2$

$$\begin{aligned} \text{R.S.} &= (\sin \theta + \cos \theta)^2 \\ &= \sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta \\ &= 1 + \sin 2\theta \\ &= \text{L.S.} \end{aligned}$$

The left side is equal to the right side, so the identity is proved.

c) $2 \cot \theta = \frac{\sin 2\theta}{1 - \cos^2 \theta}$

$$\begin{aligned} \text{R.S.} &= \frac{\sin 2\theta}{1 - \cos^2 \theta} \\ &= \frac{2 \sin \theta \cos \theta}{\sin^2 \theta} \\ &= \frac{2 \cos \theta}{\sin \theta} \\ &= 2 \cot \theta \\ &= \text{L.S.} \end{aligned}$$

The left side is equal to the right side, so the identity is proved.

d) $\tan 2\theta \cos 2\theta = 2 \sin \theta \cos \theta$

$$\begin{aligned} \text{L.S.} &= \tan 2\theta \cos 2\theta \\ &= \left(\frac{\sin 2\theta}{\cos 2\theta}\right) \cos 2\theta \\ &= \sin 2\theta \\ &= 2 \sin \theta \cos \theta \\ &= \text{R.S.} \end{aligned}$$

The left side is equal to the right side, so the identity is proved.

11. Prove each identity.

$$\text{a) } \frac{\cos 2\theta - 2 \cos^2 \theta}{2 \cos \theta} = \frac{-\sin 2\theta \sec^2 \theta}{4 \sin \theta}$$

$$\begin{aligned} \text{L.S.} &= \frac{\cos 2\theta - 2 \cos^2 \theta}{2 \cos \theta} \\ &= \frac{2 \cos^2 \theta - 1 - 2 \cos^2 \theta}{2 \cos \theta} \end{aligned}$$

$$= \frac{-1}{2 \cos \theta}$$

$$\begin{aligned} \text{R.S.} &= \frac{-\sin 2\theta \sec^2 \theta}{4 \sin \theta} \\ &= \frac{-2 \sin \theta \cos \theta}{4 \sin \theta} \end{aligned}$$

$$= \frac{-1}{2 \cos \theta}$$

$$= \frac{-1}{2 \cos \theta}$$

The left side and the right side simplify to the same expression, so the identity is proved.

$$\text{b) } \frac{\sin 2\theta}{\cos 2\theta - 1} = -\cot \theta$$

$$\begin{aligned} \text{L.S.} &= \frac{\sin 2\theta}{\cos 2\theta - 1} \\ &= \frac{2 \sin \theta \cos \theta}{1 - 2 \sin^2 \theta - 1} \end{aligned}$$

$$= \frac{2 \sin \theta \cos \theta}{-2 \sin^2 \theta}$$

$$= -\frac{\cos \theta}{\sin \theta}$$

$$= -\cot \theta$$

$$= \text{R.S.}$$

The left side is equal to the right side, so the identity is proved.

$$\text{c) } \frac{2 \cos^2 \theta}{\sin 2\theta} = \frac{1 + \cos \theta}{\tan \theta + \sin \theta}$$

$$\begin{aligned} \text{L.S.} &= \frac{2 \cos^2 \theta}{\sin 2\theta} \\ &= \frac{2 \cos \theta \cos \theta}{2 \sin \theta \cos \theta} \end{aligned}$$

$$= \frac{\cos \theta}{\sin \theta}$$

$$\begin{aligned} \text{R.S.} &= \frac{1 + \cos \theta}{\tan \theta + \sin \theta} \\ &= \frac{1 + \cos \theta}{\frac{\sin \theta}{\cos \theta} + \sin \theta} \end{aligned}$$

$$= \frac{(1 + \cos \theta)(\cos \theta)}{\sin \theta + \sin \theta \cos \theta}$$

$$= \frac{(1 + \cos \theta)(\cos \theta)}{(\sin \theta)(1 + \cos \theta)}$$

$$= \frac{\cos \theta}{\sin \theta}$$

The left side and the right side simplify to the same expression, so the identity is proved.

$$\text{d) } \frac{\sin 2\theta}{2 + 2 \cos 2\theta} = \frac{\sec^2 \theta - 1}{2 \tan \theta}$$

$$\begin{aligned} \text{L.S.} &= \frac{\sin 2\theta}{2 + 2 \cos 2\theta} \\ &= \frac{2 \sin \theta \cos \theta}{2(1 + \cos 2\theta)} \end{aligned}$$

$$= \frac{\sin \theta \cos \theta}{1 + 2 \cos^2 \theta - 1}$$

$$= \frac{\sin \theta \cos \theta}{2 \cos \theta \cos \theta}$$

$$= \frac{\sin \theta}{2 \cos \theta}$$

$$= \frac{1}{2} \tan \theta$$

$$\text{R.S.} = \frac{\sec^2 \theta - 1}{2 \tan \theta}$$

$$= \frac{\tan^2 \theta}{2 \tan \theta}$$

$$= \frac{1}{2} \tan \theta$$

The left side and right side simplify to the same expression, so the identity is proved.

12. Solve each equation over the domain $-180^\circ \leq x < 180^\circ$.

a) $\cos 2x = 2 \cos x - 1$

$$2 \cos^2 x - 1 - 2 \cos x + 1 = 0$$

$$2 \cos^2 x - 2 \cos x = 0$$

$$(2 \cos x)(\cos x - 1) = 0$$

Either $2 \cos x = 0$

$$\cos x = 0$$

$$x = 90^\circ \text{ or } x = -90^\circ$$

Or $\cos x - 1 = 0$

$$\cos x = 1$$

$$x = 0^\circ$$

The roots are: $x = 0^\circ$,

$$x = 90^\circ, \text{ and } x = -90^\circ$$

b) $\sin x = \cos 2x$

$$\sin x = 1 - 2 \sin^2 x$$

$$2 \sin^2 x + \sin x - 1 = 0$$

$$(2 \sin x - 1)(\sin x + 1) = 0$$

Either $2 \sin x - 1 = 0$

$$\sin x = \frac{1}{2}$$

$$x = 30^\circ \text{ or } x = 150^\circ$$

Or $\sin x + 1 = 0$

$$\sin x = -1$$

$$x = -90^\circ$$

The roots are: $x = 30^\circ$,

$$x = 150^\circ, \text{ and } x = -90^\circ$$

13. Solve each equation over the domain $-2\pi < x < 2\pi$.

a) $\sqrt{2} \sin 2x - 2 \sin x = 0$

$$\sqrt{2}(2 \sin x \cos x) - 2 \sin x = 0$$

$$2 \sin x (\sqrt{2} \cos x - 1) = 0$$

Either $2 \sin x = 0$

$$\sin x = 0$$

$$x = 0 \text{ or } x = \pi \text{ or } x = -\pi$$

Or $\sqrt{2} \cos x - 1 = 0$

$$\cos x = \frac{1}{\sqrt{2}}$$

$$x = \frac{\pi}{4} \text{ or } x = \frac{7\pi}{4} \text{ or } x = -\frac{\pi}{4} \text{ or}$$

$$x = -\frac{7\pi}{4}$$

The roots are: $x = 0$,

$$x = \pm \frac{\pi}{4}, x = \pm \pi, \text{ and } x = \pm \frac{7\pi}{4}$$

b) $3 \sin^2 x + \cos 2x - 2 = 0$

$$3 \sin^2 x + 1 - 2 \sin^2 x - 2 = 0$$

$$\sin^2 x - 1 = 0$$

$$\sin x = \pm 1$$

$$x = \frac{\pi}{2} \text{ or } x = \frac{3\pi}{2} \text{ or } x = -\frac{\pi}{2} \text{ or}$$

$$x = -\frac{3\pi}{2}$$

The roots are: $x = \pm \frac{\pi}{2}$ and

$$x = \pm \frac{3\pi}{2}$$

14. For this solution of the equation $\sin 2x = \cos x$ over the domain $0 \leq x < 2\pi$, identify the error then write a correct solution.

$$\sin 2x = \cos x$$

$$2 \sin x \cos x = \cos x$$

$$\frac{2 \sin x \cancel{\cos x}^1}{\cancel{\cos x}^1} = \frac{\cancel{\cos x}^1}{\cancel{\cos x}^1}$$

$$2 \sin x = 1$$

$$\sin x = \frac{1}{2}$$

$$x = \frac{\pi}{6} \text{ or } x = \frac{5\pi}{6}$$

$$\sin 2x = \cos x$$

$$2 \sin x \cos x - \cos x = 0$$

$$\cos x (2 \sin x - 1) = 0$$

Either $\cos x = 0$

$$x = \frac{\pi}{2} \text{ or } x = \frac{3\pi}{2}$$

Or $2 \sin x = 1$

$$\sin x = \frac{1}{2}$$

$$x = \frac{\pi}{6} \text{ or } x = \frac{5\pi}{6}$$

The roots are: $x = \frac{\pi}{6}, x = \frac{\pi}{2},$

$$x = \frac{5\pi}{6}, \text{ and } x = \frac{3\pi}{2}$$

In the 3rd line, $\cos x$ cannot be removed, because a possible solution is $\cos x = 0$.

15. Use algebra to solve the equation $\cos 2x = 2 \cos x$ over the set of real numbers. Give the answer to the nearest hundredth.

$$\cos 2x = 2 \cos x$$

$$2 \cos^2 x - 1 = 2 \cos x$$

$$2 \cos^2 x - 2 \cos x - 1 = 0$$

Use: $\cos x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Substitute: $a = 2, b = -2, c = -1$

$$\cos x = \frac{2 \pm \sqrt{(-2)^2 - 4(2)(-1)}}{2(2)}$$

$$\cos x = \frac{2 \pm \sqrt{12}}{4}$$

$$\cos x = \frac{2 \pm 2\sqrt{3}}{4}$$

$$\cos x = \frac{1 \pm \sqrt{3}}{2}$$

$\frac{1 + \sqrt{3}}{2} > 1$, so there is no solution for $\cos x = \frac{1 + \sqrt{3}}{2}$.

So, $\cos x = \frac{1 - \sqrt{3}}{2}$

Since $\frac{1 - \sqrt{3}}{2}$ is negative, the terminal arm of angle x lies in

Quadrant 2 or 3.

The reference angle is: $\cos^{-1}\left(\frac{\sqrt{3} - 1}{2}\right) = 1.1960\dots$

In Quadrant 2, $x = \pi - 1.1960\dots$, or $1.9455\dots$

In Quadrant 3, $x = \pi + 1.1960\dots$, or $4.3376\dots$

Verify by substituting each root in the given equation.

The solution is: $x \doteq 1.95 + 2\pi k, k \in \mathbb{Z}$ or $x \doteq 4.34 + 2\pi k, k \in \mathbb{Z}$

16. A student said that the identity $\sin 2\theta = 2 \sin \theta \cos \theta$ could be extended so that $\sin 4\theta = 4 \sin \theta \cos \theta$ and $\sin 6\theta = 6 \sin \theta \cos \theta$.

- a) Is the student correct? If your answer is yes, explain why. If your answer is no, write correct identities for $\sin 4\theta$ and $\sin 6\theta$.

$\sin 4\theta = 4 \sin \theta \cos \theta$ is not an identity.

For example, suppose: $\theta = \frac{\pi}{3}$

$$\begin{aligned} \sin 4\theta &= \sin \frac{4\pi}{3} & 4 \sin \theta \cos \theta &= 4 \sin \frac{\pi}{3} \cos \frac{\pi}{3} \\ &= -\frac{\sqrt{3}}{2} & &= 4 \left(\frac{\sqrt{3}}{2} \right) \left(\frac{1}{2} \right) \\ & & &= \sqrt{3} \end{aligned}$$

$\sin 6\theta = 6 \sin \theta \cos \theta$ is not an identity.

For example, suppose: $\theta = \frac{\pi}{3}$

$$\begin{aligned} \sin 6\theta &= \sin \frac{6\pi}{3} & 6 \sin \theta \cos \theta &= 6 \sin \frac{\pi}{3} \cos \frac{\pi}{3} \\ &= \sin 2\pi & &= 6 \left(\frac{\sqrt{3}}{2} \right) \left(\frac{1}{2} \right) \\ &= 0 & &= \frac{3\sqrt{3}}{2} \end{aligned}$$

In $\sin 2\theta = 2 \sin \theta \cos \theta$, replace θ with 2θ :

$$\sin 4\theta = 2 \sin 2\theta \cos 2\theta$$

And, in $\sin 2\theta = 2 \sin \theta \cos \theta$, replace θ with 3θ :

$$\sin 6\theta = 2 \sin 3\theta \cos 3\theta$$

- b) Write an identity for $\sin b\theta$, where b is a positive even number.

In $\sin 2\theta = 2 \sin \theta \cos \theta$, replace 2θ with $b\theta$, and replace θ with $\frac{b}{2}\theta$:

$$\sin b\theta = 2 \sin \frac{b}{2}\theta \cos \frac{b}{2}\theta$$

C

17. Prove each identity.

a) $-\sec 2x = \frac{\tan x + \cot x}{\tan x - \cot x}$

$$\begin{aligned} \text{R.S.} &= \frac{\tan x + \cot x}{\tan x - \cot x} \\ &= \frac{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}}{\frac{\sin x}{\cos x} - \frac{\cos x}{\sin x}} \\ &= \frac{\frac{\sin^2 x + \cos^2 x}{\sin x \cos x}}{\frac{\sin^2 x - \cos^2 x}{\sin x \cos x}} \\ &= \frac{1}{\frac{\sin^2 x - \cos^2 x}{\sin x \cos x}} \\ &= \frac{1}{-\cos 2x} \\ &= -\sec 2x \\ &= \text{L.S.} \end{aligned}$$

Since the left side is equal to the right side, the identity is proved.

b) $\cot 2x = \frac{\cot x - \tan x}{2}$

$$\begin{aligned} \text{R.S.} &= \frac{\cot x - \tan x}{2} \\ &= \frac{\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}}{2} \\ &= \frac{\frac{\cos^2 x - \sin^2 x}{\sin x \cos x}}{2} \\ &= \frac{\cos^2 x - \sin^2 x}{2 \sin x \cos x} \\ &= \frac{\cos 2x}{\sin 2x} \\ &= \cot 2x \\ &= \text{L.S.} \end{aligned}$$

Since the left side is equal to the right side, the identity is proved.

18. For each equation, determine the solution over the set of real numbers.

a) $4 \sin 3x \cos 3x = 1$

$$2(2 \sin 3x \cos 3x) = 1$$

$$2 \sin 2(3x) = 1$$

$$2 \sin 6x = 1$$

$$\sin 6x = \frac{1}{2}$$

$$6x = \frac{\pi}{6} + 2\pi k, k \in \mathbb{Z}, \text{ or}$$

$$6x = \frac{5\pi}{6} + 2\pi k, k \in \mathbb{Z}$$

$$\text{So, } x = \frac{\pi}{36} + \frac{\pi}{3}k, k \in \mathbb{Z}, \text{ or}$$

$$x = \frac{5\pi}{36} + \frac{\pi}{3}k, k \in \mathbb{Z}$$

b) $4 \cos^2 5x - 2 + \sqrt{3} = 0$

$$2(2 \cos^2 5x - 1) = -\sqrt{3}$$

$$2 \cos 2(5x) = -\sqrt{3}$$

$$\cos 10x = -\frac{\sqrt{3}}{2}$$

$$10x = \frac{5\pi}{6} + 2\pi k, k \in \mathbb{Z}, \text{ or}$$

$$10x = \frac{7\pi}{6} + 2\pi k, k \in \mathbb{Z}$$

$$\text{So, } x = \frac{\pi}{12} + \frac{\pi}{5}k, k \in \mathbb{Z}, \text{ or}$$

$$x = \frac{7\pi}{60} + \frac{\pi}{5}k, k \in \mathbb{Z}$$