REVIEW, pages 671–679

7.1

1. Use graphing technology to solve each equation over the given domain. Give the roots to the nearest hundredth, then determine the general solution.

a) $5 \cos 2x + 1 = 3 \sin x$; for $0 \leq x < 2\pi$

Graph $y = 5 \cos 2x + 1 - 3 \sin x$.
To the nearest hundredth, the roots are:
$x = 0.69$, $x = 2.45$, $x = 4.36$, and $x = 5.06$
The period is 2π, so the general solution is:
$x \doteq 0.69 + 2\pi k, k \in \mathbb{Z}$ or
$x \doteq 2.45 + 2\pi k, k \in \mathbb{Z}$ or
$x \doteq 4.36 + 2\pi k, k \in \mathbb{Z}$ or
$x \doteq 5.06 + 2\pi k, k \in \mathbb{Z}$

b) $3 \sin^2 x + 2 = \tan x$; for $-2\pi < x < 2\pi$

Graph $y = 3 \sin^2 x + 2 - \tan x$.
To the nearest hundredth, the roots are:
$x = -4.91$, $x = -1.77$, $x = 1.37$, and $x = 4.51$
The period is π, so the general solution is: $x \doteq 1.37 + \pi k, k \in \mathbb{Z}$

c) $2 \cos^2 x = 2x$; over the set of real numbers

Graph $y = 2 \cos^2 x - 2x$.
To the nearest hundredth, the root is: $x = 0.64$
This root is also the general solution.

2. The first two positive roots of the equation $\sin 3x = \frac{2}{5}$ are $x \doteq 0.14$ and $x \doteq 0.91$. Determine the general solution of this equation.

The period of the function is: $\frac{2\pi}{3}$
The graph of $y = \sin 3x - \frac{2}{5}$ indicates that the two given roots are the only zeros of the function in one cycle.
So, the general solution is:
$x \doteq 0.14 + \frac{2\pi}{3}k, k \in \mathbb{Z}$ or $x \doteq 0.91 + \frac{2\pi}{3}k, k \in \mathbb{Z}$

7.2

3. Verify that each given value of x is a root of the equation.

a) $\sin x + \sin^2 x = 0; x = 270°$

Substitute $x = 270°$ in the left side of the equation.

$$\begin{aligned} \text{L.S.} &= \sin 270° + (\sin 270°)^2 \\ &= (-1) + (-1)^2 \\ &= 0 \\ &= \text{R.S.} \end{aligned}$$

Since the left side is equal to the right side, the root is verified.

b) $3 \tan x - 1 = 2 \tan^2 x; x = \frac{5\pi}{4}$

Substitute $x = \frac{5\pi}{4}$ in each side of equation.

$$\begin{aligned} \text{L.S.} &= 3 \tan \frac{5\pi}{4} - 1 \\ &= 3(1)^2 - 1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \text{R.S.} &= 2 \tan^2 x \\ &= 2\left(\tan \frac{5\pi}{4}\right)^2 \\ &= 2(1)^2 \\ &= 2 \end{aligned}$$

Since the left side is equal to the right side, the root is verified.

4. Use algebra to solve each equation over the given domain, and write the general solution. Write the answers to the nearest degree or the nearest hundredth of a radian.

a) $1 + 5 \cos 2x = 0$; for $-90° \leq x \leq 180°$

$$\begin{aligned} 5 \cos 2x &= -1 \\ \cos 2x &= -\frac{1}{5} \end{aligned}$$

The reference angle is:

$\cos^{-1}\left(\frac{1}{5}\right) = 78.4630...°$

In Quadrant 2,

$2x = 180° - 78.4630...°$

$x = 50.7684...°$

In Quadrant 3,

$2x = 180° + 78.4630...°$

$x = 129.2315...°$

The period of $\cos 2x$ is 180°, so the general solution is:

$x \doteq 51° + k180°, k \in \mathbb{Z}$ or

$x \doteq 129° + k180°, k \in \mathbb{Z}$

In the given domain, the roots are:

$x \doteq 51°$, $x \doteq 129°$, and $x \doteq -51°$

b) $10 = 5 - 3 \csc x$; for $-\frac{3\pi}{2} < x < \frac{\pi}{2}$

$$\begin{aligned} 3 \csc x &= -5 \\ \csc x &= -\frac{5}{3} \\ \sin x &= -\frac{3}{5} \end{aligned}$$

The reference angle is:

$\sin^{-1}\left(\frac{3}{5}\right) = 0.6435 ...$

In Quadrant 3,

$x = -\pi + 0.6435...$

$x = -2.4980...$

In Quadrant 4,

$x = -0.6435...$

The period of $\sin x$ is 2π, so the general solution is:

$x \doteq -2.50 + 2\pi k, k \in \mathbb{Z}$ or

$x \doteq -0.64 + 2\pi k, k \in \mathbb{Z}$

In the given domain, the roots are:

$x \doteq -2.50$ and $x \doteq -0.64$

5. Use algebra to solve the equation $3 - 6\sin^2 x = -\sin x$ over the domain $0 \leq x < 2\pi$. Give the roots to the nearest hundredth.

$6\sin^2 x - \sin x - 3 = 0$

Use: $\sin x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Substitute: $a = 6, b = -1, c = -3$

$\sin x = \dfrac{1 \pm \sqrt{(-1)^2 - 4(6)(-3)}}{2(6)}$

$\sin x = \dfrac{1 \pm \sqrt{73}}{12}$

Either $\sin x = \dfrac{1 + \sqrt{73}}{12}$

$\sin x$ is positive when the terminal arm of angle x lies in Quadrant 1 or 2.

The reference angle is:

$\sin^{-1}\left(\dfrac{1 + \sqrt{73}}{12}\right) = 0.9195\ldots$

In Quadrant 1,
$x = 0.9195\ldots$

In Quadrant 2,
$x = \pi - 0.9195\ldots$
$x = 2.2220\ldots$

Or $\sin x = \dfrac{1 - \sqrt{73}}{12}$

$\sin x$ is negative when the terminal arm of angle x lies in Quadrant 3 or 4.

The reference angle is:

$\sin^{-1}\left(\dfrac{\sqrt{73} - 1}{12}\right) = 0.6798\ldots$

In Quadrant 3,
$x = \pi + 0.6798\ldots$
$x = 3.8214\ldots$

In Quadrant 4,
$x = 2\pi - 0.6798\ldots$
$x = 5.6033\ldots$

Verify by substituting each root in the given equation.
To the nearest hundredth, the roots are:
$x = 0.92$, $x = 2.22$, $x = 3.82$, and $x = 5.60$

6. Determine the general solution of the equation $1 - 3\cos x + 2\cos^2 x = 0$ over the set of real numbers.

$2\cos^2 x - 3\cos x + 1 = 0$
$(2\cos x - 1)(\cos x - 1) = 0$

Either $2\cos x - 1 = 0$

$\cos x = \dfrac{1}{2}$

In Quadrant 1, $x = \dfrac{\pi}{3}$

In Quadrant 4, $x = \dfrac{5\pi}{3}$

Or $\cos x - 1 = 0$

$\cos x = 1$

The terminal arm of angle x lies on the positive x-axis, so $x = 0$ or $x = 2\pi$

The period of the function is 2π, so the general solution is:

$x = 2\pi k, k \in \mathbb{Z}$ or $x = \dfrac{\pi}{3} + 2\pi k, k \in \mathbb{Z}$ or $x = \dfrac{5\pi}{3} + 2\pi k, k \in \mathbb{Z}$

7.3

7. For each identity:

i) Determine the non-permissible values of θ.

ii) Prove the identity.

a) $\dfrac{1 + \tan\theta}{1 + \cot\theta} = \dfrac{1 - \tan\theta}{\cot\theta - 1}$

i) Non-permissible values occur when: $\cos\theta = 0$, $\theta = \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$

$\sin\theta = 0, \theta = \pi k, k \in \mathbb{Z}$

$\cot\theta = \pm 1, \theta = \frac{\pi}{4}k, k \in \mathbb{Z}$

The non-permissible values can be written as:

$\theta = \frac{\pi}{4}k, k \in \mathbb{Z}$

ii) $$\text{L.S.} = \frac{1 + \tan\theta}{1 + \cot\theta} = \frac{1 + \tan\theta}{1 + \frac{1}{\tan\theta}} = \frac{(\tan\theta)(1 + \tan\theta)}{\tan\theta + 1} = \tan\theta$$

$$\text{R.S.} = \frac{1 - \tan\theta}{\cot\theta - 1} = \frac{1 - \tan\theta}{\frac{1}{\tan\theta} - 1} = \frac{(1 - \tan\theta)(\tan\theta)}{1 - \tan\theta} = \tan\theta$$

The left side and the right side simplify to the same expression, so the identity is proved.

b) $\dfrac{\csc^2\theta + 1}{\cot^2\theta + \cos^2\theta} = \tan\theta\sec\theta\csc\theta$

i) Non-permissible values occur when: $\cos\theta = 0$, $\theta = \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$

$\sin\theta = 0, \theta = \pi k, k \in \mathbb{Z}$

The non-permissible values can be written as: $\theta = \frac{\pi}{2}k, k \in \mathbb{Z}$

ii) $$\text{L.S.} = \frac{\csc^2\theta + 1}{\cot^2\theta + \cos^2\theta} = \frac{\frac{1}{\sin^2\theta} + 1}{\frac{\cos^2\theta}{\sin^2\theta} + \cos^2\theta} = \frac{1 + \sin^2\theta}{\cos^2\theta + \cos^2\theta\sin^2\theta} = \frac{1 + \sin^2\theta}{(\cos^2\theta)(1 + \sin^2\theta)} = \frac{1}{\cos^2\theta}$$

$$\text{R.S.} = \tan\theta\sec\theta\csc\theta = \left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{1}{\cos\theta}\right)\left(\frac{1}{\sin\theta}\right) = \frac{1}{\cos^2\theta}$$

The left side and the right side simplify to the same expression, so the identity is proved.

Home Quit

8. Use algebra to solve each equation over the given domain. Give the roots to the nearest degree.

a) $3 \tan x = 1 + 4 \cot x$; for $-180° < x < 180°$

$3 \tan x = 1 + \frac{4}{\tan x}$

$3 \tan^2 x - \tan x - 4 = 0$

$(3 \tan x - 4)(\tan x + 1) = 0$

For $3 \tan x - 4 = 0$

$\tan x = \frac{4}{3}$

The reference angle is:

$\tan^{-1}\left(\frac{4}{3}\right) = 53.1301...°$

In Quadrant 1,
$x = 53.1301...°$

In Quadrant 3,
$x = -180° + 53.1301...°$
$x = -126.8698...°$

For $\tan x + 1 = 0$

$\tan x = -1$

The reference angle is:
$\tan^{-1} 1 = 45°$

In Quadrant 2,
$x = 180° - 45°$
$x = 135°$

In Quadrant 4,
$x = -45°$

The roots are: $x \doteq -127°$, $x = -45°$, $x \doteq 53°$, and $x = 135°$

b) $5 = \csc x - 6 \sin x$; for $-90° \leq x < 270°$

$5 = \frac{1}{\sin x} - 6 \sin x$

$6 \sin^2 x + 5 \sin x - 1 = 0$

$(6 \sin x - 1)(\sin x + 1) = 0$

For $6 \sin x - 1 = 0$

$6 \sin x = 1$

$\sin x = \frac{1}{6}$

The reference angle is:

$\sin^{-1}\left(\frac{1}{6}\right) = 9.5940...°$

In Quadrant 1,
$x = 9.5940...°$

In Quadrant 2,
$x = 180° - 9.5940...°$
$x = 170.4059...°$

For $\sin x + 1 = 0$

$\sin x = -1$

The terminal arm of angle x lies on the negative y-axis.
So, $x = -90°$

The roots are: $x = -90°$, $x \doteq 10°$, and $x \doteq 170°$

7.4

9. For each identity:

i) Verify the identity for $\theta = \frac{11\pi}{6}$.

ii) Prove the identity.

a) $\frac{1 - \cos\theta}{\sin^2\theta} = \frac{1}{1 + \cos\theta}$

i) Substitute $\theta = \frac{11\pi}{6}$ in each side of the identity.

$$\text{L.S.} = \frac{1 - \cos\theta}{\sin^2\theta} = \frac{1 - \cos\frac{11\pi}{6}}{\left(\sin\frac{11\pi}{6}\right)^2} = \frac{1 - \frac{\sqrt{3}}{2}}{\left(-\frac{1}{2}\right)^2} = 2(2 - \sqrt{3})$$

$$\text{R.S.} = \frac{1}{1 + \cos\theta} = \frac{1}{1 + \cos\frac{11\pi}{6}} = \frac{1}{1 + \frac{\sqrt{3}}{2}} = \frac{2}{2 + \sqrt{3}} = \frac{2}{2 + \sqrt{3}} \cdot \frac{2 - \sqrt{3}}{2 - \sqrt{3}} = 2(2 - \sqrt{3})$$

Since the left side is equal to the right side, the identity is verified.

ii)
$$\text{L.S.} = \frac{1 - \cos\theta}{\sin^2\theta} = \frac{1 - \cos\theta}{1 - \cos^2\theta} = \frac{1 - \cos\theta}{(1 - \cos\theta)(1 + \cos\theta)} = \frac{1}{1 + \cos\theta} = \text{R.S.}$$

The left side is equal to the right side, so the identity is proved.

b) $\frac{1}{1 + \sin\theta} = \sec^2\theta - \frac{\tan\theta}{\cos\theta}$

i) Substitute $\theta = \frac{11\pi}{6}$ in each side of the identity.

$$\text{L.S.} = \frac{1}{1 + \sin\theta} = \frac{1}{1 + \sin\frac{11\pi}{6}} = \frac{1}{1 - \frac{1}{2}} = 2$$

$$\text{R.S.} = \sec^2\theta - \frac{\tan\theta}{\cos\theta} = \left(\sec\frac{11\pi}{6}\right)^2 - \frac{\tan\frac{11\pi}{6}}{\cos\frac{11\pi}{6}} = \left(\frac{2}{\sqrt{3}}\right)^2 - \frac{-\frac{1}{\sqrt{3}}}{\frac{\sqrt{3}}{2}} = \frac{4}{3} + \frac{2}{3} = 2$$

Since the left side is equal to the right side, the identity is verified.

ii)
$$\text{R.S.} = \sec^2\theta - \frac{\tan\theta}{\cos\theta} = \frac{1}{\cos^2\theta} - \frac{\sin\theta}{\cos^2\theta} = \frac{1 - \sin\theta}{1 - \sin^2\theta} = \frac{1 - \sin\theta}{(1 - \sin\theta)(1 + \sin\theta)} = \frac{1}{1 + \sin\theta} = \text{L.S.}$$

The left side is equal to the right side, so the identity is proved.

10. Use algebra to solve each equation over the domain $-2\pi < x < 2\pi$. Give the roots to the nearest hundredth where necessary.

a) $2\sec^2 x - 2 = 3 + \tan x$

$2(\sec^2 x - 1) = 3 + \tan x$
$2\tan^2 x - \tan x - 3 = 0$
$(2\tan x - 3)(\tan x + 1) = 0$
For $2\tan x - 3 = 0$
$\tan x = \frac{3}{2}$
The reference angle is:
$\tan^{-1}\left(\frac{3}{2}\right) = 0.9827...$
In Quadrant 1,
$x = 0.9827...$
or $x = -2\pi + 0.9827...$
$x = -5.3003...$
In Quadrant 3,
$x = \pi + 0.9827...$
$x = 4.1243...$
or $x = -\pi + 0.9827...$
$x = -2.1587...$
For $\tan x + 1 = 0$
$\tan x = -1$
The reference angle is:
$\tan^{-1} 1 = \frac{\pi}{4}$
In Quadrant 2,
$x = \pi - \frac{\pi}{4}$
$x = \frac{3\pi}{4}$
or $x = -\pi - \frac{\pi}{4}$
$x = -\frac{5\pi}{4}$
In Quadrant 4,
$x = 2\pi - \frac{\pi}{4}$
$x = \frac{7\pi}{4}$
or $x = -\frac{\pi}{4}$
The roots are: $x \doteq -5.30$, $x = -\frac{5\pi}{4}$, $x \doteq -2.16$, $x = -\frac{\pi}{4}$, $x \doteq 0.98$, $x = \frac{3\pi}{4}$, $x \doteq 4.12$, and $x = \frac{7\pi}{4}$

b) $4\sin x\cos x = 2 - 2\cos^2 x$

$4\sin x\cos x = 2(1 - \cos^2 x)$
$2\sin^2 x - 4\sin x\cos x = 0$
$(2\sin x)(\sin x - 2\cos x) = 0$
For $2\sin x = 0$
$\sin x = 0$
$x = 0$ or $x = \pm\pi$
For $\sin x - 2\cos x = 0$
$\sin x = 2\cos x$
$\tan x = 2$, $\cos x \neq 0$
The reference angle is:
$\tan^{-1} 2 = 1.1071...$
In Quadrant 1,
$x = 1.1071...$
or $x = -2\pi + 1.1071...$
$x = -5.1760...$
In Quadrant 3,
$x = \pi + 1.1071...$
$x = 4.2487...$
or $x = -\pi + 1.1071...$
$x = -2.0344...$
The roots are: $x = 0$, $x = \pm\pi$, $x \doteq -5.18$, $x \doteq -2.03$, $x \doteq 1.11$, and $x \doteq 4.25$

7.5

11. Prove each identity.

a) $\sin\theta = \cos\left(\frac{3\pi}{2} + \theta\right)$

$$\text{R.S.} = \cos\left(\frac{3\pi}{2} + \theta\right)$$
$$= \cos\frac{3\pi}{2}\cos\theta - \sin\frac{3\pi}{2}\sin\theta$$
$$= (0)\cos\theta - (-1)\sin\theta$$
$$= \sin\theta$$
$$= \text{L.S.}$$

Since the left side is equal to the right side, the identity is proved.

b) $-\cos\theta = \sin\left(\frac{3\pi}{2} - \theta\right)$

$$\text{R.S.} = \sin\left(\frac{3\pi}{2} - \theta\right)$$
$$= \sin\frac{3\pi}{2}\cos\theta - \cos\frac{3\pi}{2}\sin\theta$$
$$= (-1)\cos\theta - (0)\sin\theta$$
$$= -\cos\theta$$
$$= \text{L.S.}$$

Since the left side is equal to the right side, the identity is proved.

12. Solve this equation over the domain $-90° < x < 90°$:

$$\sin x \cos 2x + \cos x \sin 2x = \frac{1}{\sqrt{2}}$$

$$\sin(x + 2x) = \frac{1}{\sqrt{2}}$$
$$\sin 3x = \frac{1}{\sqrt{2}}$$

The given domain for angle x is $-90° < x < 90°$, so the domain for angle $3x$ is $-270° < 3x < 270°$:

$3x = 45°$ or $3x = 135°$ or $3x = -225°$

$x = 15°$ $x = 45°$ $x = -75°$

The roots are: $x = 15°$, $x = 45°$, and $x = -75°$

13. Determine the exact value of tan 105°.

$$\tan 105° = \tan(60° + 45°)$$
$$= \frac{\tan 60° + \tan 45°}{1 - \tan 60° \tan 45°}$$
$$= \frac{\sqrt{3} + 1}{1 - \sqrt{3}(1)}$$
$$= \frac{\sqrt{3} + 1}{1 - \sqrt{3}}$$

7.6

14. Prove each identity.

a) $\dfrac{\cot^2\theta - 1}{\csc^2\theta} = \cos 2\theta$

$$\text{L.S.} = \frac{\cot^2\theta - 1}{\csc^2\theta}$$
$$= \frac{\frac{\cos^2\theta}{\sin^2\theta} - 1}{\frac{1}{\sin^2\theta}}$$
$$= \frac{\cos^2\theta - \sin^2\theta}{1}$$
$$= \cos 2\theta$$
$$= \text{R.S.}$$

Since the left side is equal to the right side, the identity is proved.

b) $\sin 2\theta = \dfrac{2\tan\theta}{1 + \tan^2\theta}$

$$\text{R.S.} = \frac{2\tan\theta}{1 + \tan^2\theta}$$
$$= \frac{2\tan\theta}{\sec^2\theta}$$
$$= \frac{\frac{2\sin\theta}{\cos\theta}}{\frac{1}{\cos^2\theta}}$$
$$= \frac{2\sin\theta\cos\theta}{1}$$
$$= \sin 2\theta$$
$$= \text{L.S.}$$

Since the left side is equal to the right side, the identity is proved.

15. Solve each equation over the domain $-2\pi < x < 2\pi$.

a) $\sin 2x + \sqrt{3}\cos x = 0$

$$2\sin x\cos x + \sqrt{3}\cos x = 0$$
$$(\cos x)(2\sin x + \sqrt{3}) = 0$$

For $\cos x = 0$

$$x = \pm\frac{\pi}{2} \text{ or } x = \pm\frac{3\pi}{2}$$

For $2\sin x + \sqrt{3} = 0$

$$\sin x = -\frac{\sqrt{3}}{2}$$
$$x = \frac{4\pi}{3} \text{ or } x = \frac{5\pi}{3} \text{ or}$$
$$x = -\frac{2\pi}{3} \text{ or } x = -\frac{\pi}{3}$$

The roots are: $x = \pm\frac{\pi}{2}$, $x = \pm\frac{3\pi}{2}$, $x = -\frac{2\pi}{3}$, $x = -\frac{\pi}{3}$, $x = \frac{4\pi}{3}$, and $x = \frac{5\pi}{3}$

b) $\cos 2x + 5\cos x = 4\cos^2 x - 4$

$$2\cos^2 x - 1 + 5\cos x = 4\cos^2 x - 4$$
$$2\cos^2 x - 5\cos x - 3 = 0$$
$$(2\cos x + 1)(\cos x - 3) = 0$$

For $2\cos x + 1 = 0$

$$\cos x = -\frac{1}{2}$$
$$x = \pm\frac{2\pi}{3} \text{ or } x = \pm\frac{4\pi}{3}$$

For $\cos x - 3 = 0$

$$\cos x = 3$$

This has no solution.

The roots are: $x = \pm\frac{2\pi}{3}$ and $x = \pm\frac{4\pi}{3}$