Lesson 8.6 Exercises, pages 743–749

A

3. Expand using Pascal's triangle.

a) $(x + 1)^5$

The exponent is 5, so use the terms in row 6 of Pascal's triangle as coefficients: 1, 5, 10, 10, 5, 1

$$\begin{aligned}(x + 1)^5 &= 1(x)^5 + 5(x)^4(1) + 10(x)^3(1)^2 + 10(x)^2(1)^3 + 5(x)^1(1)^4 + 1(1)^5\\ &= x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1\end{aligned}$$

b) $(x - 1)^6$

The exponent is 6, so use the terms in row 7 of Pascal's triangle as coefficients: 1, 6, 15, 20, 15, 6, 1

$$\begin{aligned}(x - 1)^6 &= 1(x)^6 + 6(x)^5(-1) + 15(x)^4(-1)^2 + 20(x)^3(-1)^3\\ &\quad + 15(x)^2(-1)^4 + 6(x)(-1)^5 + 1(-1)^6\\ &= x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1\end{aligned}$$

c) $(x + y)^4$

The exponent is 4, so use the terms in row 5 of Pascal's triangle as coefficients: 1, 4, 6, 4, 1

$$\begin{aligned}(x + y)^4 &= 1(x)^4 + 4(x)^3(y) + 6(x)^2(y)^2 + 4(x)(y)^3 + 1(y)^4\\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\end{aligned}$$

d) $(x - y)^8$

The exponent is 8, so use the terms in row 9 of Pascal's triangle as coefficients: 1, 8, 28, 56, 70, 56, 28, 8, 1

$$\begin{aligned}(x - y)^8 &= 1(x)^8 + 8(x)^7(-y) + 28(x)^6(-y)^2 + 56(x)^5(-y)^3 + 70(x)^4(-y)^4\\ &\quad + 56(x)^3(-y)^5 + 28(x)^2(-y)^6 + 8(x)(-y)^7 + 1(-y)^8\\ &= x^8 - 8x^7y + 28x^6y^2 - 56x^5y^3 + 70x^4y^4 - 56x^3y^5\\ &\quad + 28x^2y^6 - 8xy^7 + y^8\end{aligned}$$

4. Determine each missing number in the expansion of $(x + y)^7$.

$x^7 + \square x^6y + 21x^5y^2 + 35x^{\square}y^3 + \square x^3y^4 + 21x^{\square}y^{\square} + 7xy^6 + y^{\square}$

The exponent is 7, so the coefficients of the terms in the expansion are the terms in row 8 of Pascal's triangle: 1, 7, 21, 35, 35, 21, 7, 1
The exponents in each term must add to 7.
The exponents of the powers of x start at 7 and decrease by 1 each time.
The exponents of the powers of y start at 0 and increase by 1 each time.
So, the missing numbers are: 7, 4, 35, 2, 5, 7

5. Determine the indicated term in each expansion.

a) the last term in $(x + 1)^9$

The last term in the expansion of $(x + y)^n$ is y^n.
So, the last term in the expansion of $(x + 1)^9$ is 1^9, or 1.

b) the 1st term in $(x - 1)^{12}$

The first term in the expansion of $(x + y)^n$ is x^n.
So, the first term in the expansion of $(x - 1)^{12}$ is x^{12}.

B

6. **a)** Multiply 4 factors of $(x - 5)$.

$$\begin{aligned}(x-5)^4 &= (x-5)(x-5)(x-5)(x-5)\\ &= (x^2-10x+25)(x^2-10x+25)\\ &= x^4-10x^3+25x^2-10x^3+100x^2-250x+25x^2-250x+625\\ &= x^4-20x^3+150x^2-500x+625\end{aligned}$$

b) Use the binomial theorem to expand $(x - 5)^4$.

$(x+y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y + {}_nC_2x^{n-2}y^2 + \dots + {}_nC_ny^n$

Substitute: $n = 4$, $y = -5$

$$\begin{aligned}(x-5)^4 &= {}_4C_0(x)^4 + {}_4C_1(x)^{4-1}(-5) + {}_4C_2(x)^{4-2}(-5)^2\\ &\quad + {}_4C_3(x)^{4-3}(-5)^3 + {}_4C_4(-5)^4\\ &= 1(x)^4 + 4(x)^3(-5) + 6(x)^2(25) + 4(x)^1(-125) + 1(625)\\ &= x^4 - 20x^3 + 150x^2 - 500x + 625\end{aligned}$$

c) Compare the two methods. What conclusions can you make?

I find it easier to use the binomial theorem; it saves time and it is less cumbersome than multiplying 4 factors.

7. Expand using the binomial theorem.

a) $(x + 2)^6$

$(x+y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y + {}_nC_2x^{n-2}y^2 + \dots + {}_nC_ny^n$

Substitute: $n = 6$, $y = 2$

$$\begin{aligned}(x+2)^6 &= {}_6C_0(x)^6 + {}_6C_1(x)^{6-1}(2) + {}_6C_2(x)^{6-2}(2)^2 + {}_6C_3(x)^{6-3}(2)^3\\ &\quad + {}_6C_4(x)^{6-4}(2)^4 + {}_6C_5(x)^{6-5}(2)^5 + {}_6C_6(2)^6\\ &= 1(x)^6 + 6(x)^5(2) + 15(x)^4(4) + 20(x)^3(8) + 15(x)^2(16)\\ &\quad + 6(x)^1(32) + 1(64)\\ &= x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64\end{aligned}$$

b) $(x^2 - 3)^5$

$(x+y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y + {}_nC_2x^{n-2}y^2 + \dots + {}_nC_ny^n$

Substitute: $n = 5$, $x = x^2$, $y = -3$

$$\begin{aligned}(x^2-3)^5 &= {}_5C_0(x^2)^5 + {}_5C_1(x^2)^4(-3) + {}_5C_2(x^2)^3(-3)^2 + {}_5C_3(x^2)^2(-3)^3\\ &\quad + {}_5C_4(x^2)^1(-3)^4 + {}_5C_5(-3)^5\\ &= 1(x^{10}) + 5(x^2)^4(-3) + 10(x^2)^3(9) + 10(x^2)^2(-27)\\ &\quad + 5(x^2)^1(81) + 1(-243)\\ &= x^{10} - 15x^8 + 90x^6 - 270x^4 + 405x^2 - 243\end{aligned}$$

c) $(3x - 2)^4$

$(x + y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y + {}_nC_2x^{n-2}y^2 + \ldots + {}_nC_ny^n$
Substitute: $n = 4$, $x = 3x$, $y = -2$

$$\begin{aligned}(3x - 2)^4 &= {}_4C_0(3x)^4 + {}_4C_1(3x)^3(-2) + {}_4C_2(3x)^2(-2)^2 + {}_4C_3(3x)(-2)^3 \\ &\quad + {}_4C_4(-2)^4 \\ &= 1(81x^4) + 4(27x^3)(-2) + 6(9x^2)(4) + 4(3x)(-8) + 1(16) \\ &= 81x^4 - 216x^3 + 216x^2 - 96x + 16\end{aligned}$$

d) $(-2 + 2x)^4$

$(x + y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y + {}_nC_2x^{n-2}y^2 + \ldots + {}_nC_ny^n$
Substitute: $n = 4$, $x = -2$, $y = 2x$

$$\begin{aligned}(-2 + 2x)^4 &= {}_4C_0(-2)^4 + {}_4C_1(-2)^3(2x) + {}_4C_2(-2)^2(2x)^2 \\ &\quad + {}_4C_3(-2)(2x)^3 + {}_4C_4(2x)^4 \\ &= 1(16) + 4(-8)(2x) + 6(4)(4x^2) + 4(-2)(8x^3) + 1(16x^4) \\ &= 16 - 64x + 96x^2 - 64x^3 + 16x^4\end{aligned}$$

e) $(-4 + 3x^4)^5$

$(x + y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y + {}_nC_2x^{n-2}y^2 + \ldots + {}_nC_ny^n$
Substitute: $n = 5$, $x = -4$, $y = 3x^4$

$$\begin{aligned}(-4 + 3x^4)^5 &= {}_5C_0(-4)^5 + {}_5C_1(-4)^4(3x^4) + {}_5C_2(-4)^3(3x^4)^2 + {}_5C_3(-4)^2(3x^4)^3 \\ &\quad + {}_5C_4(-4)(3x^4)^4 + {}_5C_5(3x^4)^5 \\ &= 1(-1024) + 5(256)(3x^4) + 10(-64)(9x^8) + 10(16)(27x^{12}) \\ &\quad + 5(-4)(81x^{16}) + 1(243x^{20}) \\ &= -1024 + 3840x^4 - 5760x^8 + 4320x^{12} - 1620x^{16} + 243x^{20}\end{aligned}$$

8. a) Write the terms in row 7 of Pascal's triangle.

1, 6, 15, 20, 15, 6, 1

b) Use your answer to part a to write the first 3 terms in each expansion.

i) $(x - 3)^6$

The exponent is 6, so the terms in row 7 of Pascal's triangle are the coefficients of the terms in the expansion of the binomial.
So, $(x + y)^6 = 1x^6 + 6x^5y + 15x^4y^2 + \ldots$
Substitute: $y = -3$

$$\begin{aligned}(x - 3)^6 &= 1x^6 + 6x^5(-3) + 15x^4(-3)^2 + \ldots \\ &= x^6 - 18x^5 + 135x^4 + \ldots\end{aligned}$$

So, the first 3 terms in the expansion are: $x^6 - 18x^5 + 135x^4$

ii) $(a + 4b)^6$

$(x + y)^6 = 1x^6 + 6x^5y + 15x^4y^2 + \ldots$
Substitute: $x = a$, $y = 4b$
$(a + 4b)^6 = 1a^6 + 6a^5(4b) + 15a^4(4b)^2 + \ldots$
$= a^6 + 24a^5b + 240a^4b^2 + \ldots$
So, the first 3 terms in the expansion are: $a^6 + 24a^5b + 240a^4b^2$

iii) $(-2a + 1)^6$

$(x + y)^6 = 1x^6 + 6x^5y + 15x^4y^2 + \ldots$
Substitute: $x = -2a$, $y = 1$
$(-2a + 1)^6 = 1(-2a)^6 + 6(-2a)^5(1) + 15(-2a)^4(1)^2 + \ldots$
$= 64a^6 - 192a^5 + 240a^4 + \ldots$
So, the first 3 terms in the expansion are: $64a^6 - 192a^5 + 240a^4$

iv) $(2x + 5y^2)^6$

$(x + y)^6 = 1x^6 + 6x^5y + 15x^4y^2 + \ldots$
Substitute: $x = 2x$, $y = 5y^2$
$(2x + 5y^2)^6 = 1(2x)^6 + 6(2x)^5(5y^2) + 15(2x)^4(5y^2)^2 + \ldots$
$= 64x^6 + 960x^5y^2 + 6000x^4y^4 + \ldots$
So, the first 3 terms in the expansion are: $64x^6 + 960x^5y^2 + 6000x^4y^4$

9. Determine the coefficient of each term.

a) x^5 in $(x + 1)^8$

x^5 is the 4th term in $(x + 1)^8$.
The coefficient of the 4th term is:
${}_8C_{4-1}$ or ${}_8C_3$
${}_8C_3 = 56$
So, the coefficient of x^5 is 56.

b) x^9 in $(x + y)^9$

x^9 is the first term in the expansion of $(x + y)^9$.
So, the coefficient of x^9 is 1.

c) x^2y in $(x + y)^3$

The coefficients of the terms will be the terms in row 4 of Pascal's triangle: 1, 3, 3, 1
x^2y is the 2nd term in the expansion of $(x + y)^3$.
So, the coefficient of x^2y is 3.

d) x^2y^3 in $(x + y)^5$

The coefficients of the terms will be the terms in row 6 of Pascal's triangle: 1, 5, 10, 10, 5, 1
x^2y^3 is the 4th term in the expansion of $(x + y)^5$.
So, the coefficient of x^2y^3 is 10.

10. Explain why the coefficients of the 3rd term and the 3rd-last term in the expansion of $(x + y)^n$, $n \geq 2$, are the same.

Since each of x and y has coefficient 1, the coefficients of the terms in the expansion of $(x + y)^n$ correspond to the terms in row $(n + 1)$ of Pascal's triangle.
In any row of Pascal's triangle, the 3rd term and 3rd-last term are the same.

11. Determine the indicated term in each expansion.

a) the last term in $(3x + 2)^5$

The last term in the expansion of $(x + y)^n$ is y^n.
Substitute: $y = 2$, $n = 5$
$2^5 = 32$
So, the last term is 32.

b) the 1st term in $(-2x + 5)^7$

The first term in the expansion of $(x + y)^n$ is x^n.
Substitute: $x = -2x$, $n = 7$
$(-2x)^7 = -128x^7$
So, the 1st term is $-128x^7$.

c) the 2nd term in $(3x - 3)^4$

The second term in the expansion of $(x + y)^n$ is $nx^{n-1}y$.
Substitute: $x = 3x$, $y = -3$, $n = 4$
$4(3x)^3(-3) = -324x^3$
So, the 2nd term is $-324x^3$.

d) the 6th term in $(4x + 1)^8$

The kth term is: ${}_nC_{k-1}x^{n-(k-1)}y^{k-1}$
Substitute: $n = 8$, $k = 6$, $x = 4x$, $y = 1$
$${}_8C_5(4x)^3(1)^5 = 56(64x^3)(1)$$
$$= 3584x^3$$
So, the 6th term is $3584x^3$.

12. When will the coefficients of the terms in the expansion of $(ax + b)^n$ be the same as the terms in row $(n + 1)$ of Pascal's triangle?

The coefficients of the terms in the expansion of $(x + y)^n$ correspond to the terms in row $(n + 1)$ of Pascal's triangle. So, both a and b must equal 1.

13. Expand and simplify $(x + 1)^8 + (x - 1)^8$. What strategy did you use?

The coefficients of the terms in the expansion of $(x + 1)^8$ are the terms in row 9 of Pascal's triangle: 1, 8, 28, 56, 70, 56, 28, 8, 1
The coefficients of the terms in the expansion of $(x - 1)^8$ depend on whether the term number is odd or even. When the term numbers are odd, the coefficients are the terms in row 9 of Pascal's triangle. When the term numbers are even, the coefficients are the opposites of the terms in row 9.
So, the coefficients of the terms in the expansion of $(x - 1)^8$ are:
1, −8, 28, −56, 70, −56, 28, −8, 1
When the terms in the two expansions are added, every second term is eliminated as the sum of the terms is 0.
$(x + 1)^8 + (x - 1)^8 = 2x^8 + 56x^6 + 140x^4 + 56x^2 + 2$

14. a) Show that the expansion of $(-2x + 1)^6$ is the same as the expansion of $(2x - 1)^6$.

Use row 7 of Pascal's triangle: 1, 6, 15, 20, 15, 6, 1

$$\begin{aligned}(-2x + 1)^6 &= 1(-2x)^6 + 6(-2x)^5(1) + 15(-2x)^4(1)^2 + 20(-2x)^3(1)^3 \\ &\quad + 15(-2x)^2(1)^4 + 6(-2x)^1(1)^5 + (1)^6 \\ &= 64x^6 + 6(-32x^5) + 15(16x^4) + 20(-8x^3) + 15(4x^2) + 6(-2x) + 1 \\ &= 64x^6 - 192x^5 + 240x^4 - 160x^3 + 60x^2 - 12x + 1\end{aligned}$$

$$\begin{aligned}(2x - 1)^6 &= 1(2x)^6 + 6(2x)^5(-1) + 15(2x)^4(-1)^2 + 20(2x)^3(-1)^3 \\ &\quad + 15(2x)^2(-1)^4 + 6(2x)^1(-1)^5 + (-1)^6 \\ &= 64x^6 + 6(32x^5)(-1) + 15(16x^4) + 20(8x^3)(-1) + 15(4x^2) \\ &\quad + 6(2x)(-1) + 1 \\ &= 64x^6 - 192x^5 + 240x^4 - 160x^3 + 60x^2 - 12x + 1\end{aligned}$$

b) Will $(-ax + b)^n$ always have the same expansion as $(ax - b)^n$? Explain.

No, when n is odd, the expansions will not be the same. For example, look at the first terms in the expansions of $(-2x + 1)^3$ and $(2x - 1)^3$.
For $(-2x + 1)^3$: the 1st term in the expansion is: $1(-2x)^3 = -8x^3$
For $(2x - 1)^3$: the 1st term in the expansion is: $1(2x)^3 = 8x^3$
Since the 1st terms are different, the expansions are not the same.

15. Which binomial power when expanded results in
$16x^4 - 32x^3 + 24x^2 - 8x + 1$?
What strategy did you use to find out?

The first term is $16x^4$. The exponent of x is 4, so the binomial has the form $(ax + b)^4$.
The coefficient a must be $\sqrt[4]{16}$: $a = 2$, or $a = -2$
The last term is 1, so b is either 1 or -1 because $(-1)^4 = 1^4$, or 1.
The terms alternate in sign, so the coefficients a and b must have different signs.
The binomial power is either $(-2x + 1)^4$ or $(2x - 1)^4$.

16. Expand using the binomial theorem.

a) $(0.2x - 1.2y)^5$

$(x + y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y + {}_nC_2x^{n-2}y^2 + \ldots + {}_nC_ny^n$
Substitute: $n = 5$, $x = 0.2x$, $y = -1.2y$

$$\begin{aligned}(0.2x - 1.2y)^5 &= {}_5C_0(0.2x)^5 + {}_5C_1(0.2x)^4(-1.2y) + {}_5C_2(0.2x)^3(-1.2y)^2 \\ &\quad + {}_5C_3(0.2x)^2(-1.2y)^3 + {}_5C_4(0.2x)(-1.2y)^4 + {}_5C_5(-1.2y)^5 \\ &= 1(0.000\,32x^5) + 5(0.0016x^4)(-1.2y) + 10(0.008x^3)(1.44y^2) \\ &\quad + 10(0.04x^2)(-1.728y^3) + 5(0.2x)(2.0736y^4) + 1(-2.488\,32y^5) \\ &= 0.000\,32x^5 - 0.0096x^4y + 0.1152x^3y^2 - 0.6912x^2y^3 \\ &\quad + 2.0736xy^4 - 2.488\,32y^5\end{aligned}$$

b) $\left(\frac{3}{8}a + \frac{1}{6}b\right)^4$

$(x + y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y + {}_nC_2x^{n-2}y^2 + \ldots + {}_nC_ny^n$

Substitute: $n = 4$, $x = \frac{3}{8}a$, $y = \frac{1}{6}b$

$$\begin{aligned}\left(\frac{3}{8}a + \frac{1}{6}b\right)^4 &= {}_4C_0\left(\frac{3}{8}a\right)^4 + {}_4C_1\left(\frac{3}{8}a\right)^3\left(\frac{1}{6}b\right) + {}_4C_2\left(\frac{3}{8}a\right)^2\left(\frac{1}{6}b\right)^2 \\ &\quad + {}_4C_3\left(\frac{3}{8}a\right)\left(\frac{1}{6}b\right)^3 + {}_4C_4\left(\frac{1}{6}b\right)^4 \\ &= 1\left(\frac{81}{4096}a^4\right) + 4\left(\frac{27}{512}a^3\right)\left(\frac{1}{6}b\right) + 6\left(\frac{9}{64}a^2\right)\left(\frac{1}{36}b^2\right) \\ &\quad + 4\left(\frac{3}{8}a\right)\left(\frac{1}{216}b^3\right) + 1\left(\frac{1}{1296}b^4\right) \\ &= \frac{81}{4096}a^4 + \frac{27}{768}a^3b + \frac{9}{384}a^2b^2 + \frac{3}{432}ab^3 + \frac{1}{1296}b^4 \\ &= \frac{81}{4096}a^4 + \frac{9}{256}a^3b + \frac{3}{128}a^2b^2 + \frac{1}{144}ab^3 + \frac{1}{1296}b^4\end{aligned}$$

C

17. Determine the 3rd term in the expansion of $(x^2 + 2x + 1)^6$.

$(x^2 + 2x + 1) = (x + 1)^2$

So, $(x^2 + 2x + 1)^6 = ((x + 1)^2)^6$

$= (x + 1)^{12}$

The coefficients in the expansion of $(x + 1)^{12}$ are the terms in row 13 of Pascal's triangle.

The 3rd term in row 13 is 66.

So, the 3rd term in the expansion of $(x^2 + 2x + 1)^6$ is: $66(x)^{10}(1)^2$, or $66x^{10}$

18. a) Show that ${}_nC_0 + {}_nC_1 + {}_nC_2 + \ldots + {}_nC_{n-1} + {}_nC_n = 2^n$

Express 2^n as the binomial $(1 + 1)^n$ and expand:

$(1 + 1)^n = {}_nC_0(1)^n + {}_nC_1(1)^{n-1}(1) + {}_nC_2(1)^{n-2}(1)^2 + \ldots$
$+ {}_nC_{n-1}(1)(1)^{n-1} + {}_nC_n(1)^n$

So, $2^n = {}_nC_0 + {}_nC_1 + {}_nC_2 + \ldots + {}_nC_{n-1} + {}_nC_n$

b) What does the relationship in part a indicate about the sum of the terms in any row of Pascal's triangle?

${}_nC_0, {}_nC_1, {}_nC_2, \ldots {}_nC_{n-1}, {}_nC_n$ are the terms in row $(n + 1)$ of the triangle.

From part a, the sum of these terms is 2^n.

So, the sum of the terms in any row of Pascal's triangle is a power of 2.