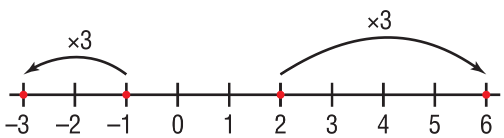
Connect

We can use a number line to investigate the effect of multiplying and dividing each side of an inequality by the same number.

Consider the inequality: $-1 < 2$

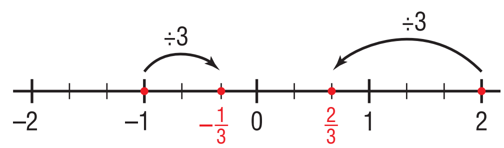
Multiply each side by 3.

$$-1 < 2$$
$$(-1)(3) < (2)(3)$$
$$-3 < 6$$



Divide each side by 3.

$$-1 < 2$$
$$(-1) \div 3 < 2 \div 3$$
$$-\frac{1}{3} < \frac{2}{3}$$



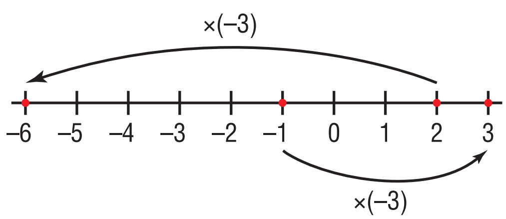
Multiply each side by -3.

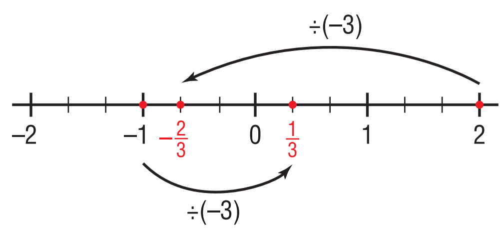
$$-1 < 2$$
$$(-1)(-3) > (2)(-3)$$
$$3 > -6$$

We must reverse the inequality sign for each inequality to remain true.

Divide each side by -3.

$$-1 < 2$$
$$(-1) \div (-3) > 2 \div (-3)$$
$$\frac{1}{3} > -\frac{2}{3}$$





The examples above illustrate these properties of inequalities:

- When each side of an inequality is multiplied or divided by the same positive number, the resulting inequality is still true.
- When each side of an inequality is multiplied or divided by the same negative number, the inequality sign must be reversed for the inequality to remain true.

To solve an inequality, we use the same strategy as for solving an equation. However, when we multiply or divide by a negative number, we reverse the inequality sign.

Example 1 Solving a One-step Inequality

Solve each inequality. Graph each solution.

a) $-5s \leq 25$

b) $7a < -21$

c) $\frac{y}{-4} > -3$

d) $\frac{k}{3} \geq -2$

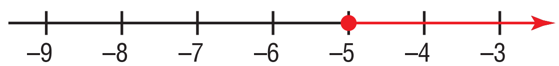
A Solution

a) $-5s \leq 25$

As you divide each side by -5, reverse the inequality sign.

$$\frac{-5s}{-5} \geq \frac{25}{-5}$$

$$s \geq -5$$

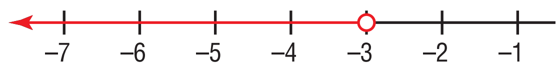


b) $7a < -21$

Divide each side by 7.

$$\frac{7a}{7} < \frac{-21}{7}$$

$$a < -3$$

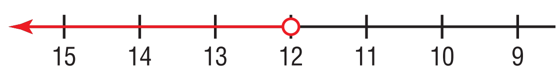


c) $\frac{y}{-4} > -3$

As you multiply each side by -4, reverse the inequality sign.

$$-4\left(\frac{y}{-4}\right) < -4\,(-3)$$

$$y < 12$$

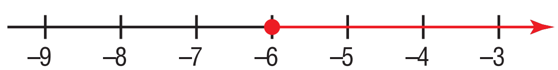


d) $\frac{k}{3} \geq -2$

Multiply each side by 3.

$$3\left(\frac{k}{3}\right) \geq 3(-2)$$

$$k \geq -6$$



Example 2 Solving a Multi-Step Inequality

a) Solve this inequality: $-2.6a + 14.6 > -5.2 + 1.8a$

b) Verify the solution.

A Solution

a)

$$-2.6a + 14.6 > -5.2 + 1.8a$$ Subtract 14.6 from each side.

$$-2.6a + 14.6 - 14.6 > -5.2 - 14.6 + 1.8a$$

$$-2.6a > -19.8 + 1.8a$$ Subtract $1.8a$ from each side.

$$-2.6a - 1.8a > -19.8 + 1.8a - 1.8a$$

$$-4.4a > -19.8$$ Divide each side by -4.4 and reverse the inequality sign.

$$\frac{-4.4a}{-4.4} < \frac{-19.8}{-4.4}$$

$$a < 4.5$$

b) The solution of the inequality $a < 4.5$ is all numbers less than 4.5.
Choose several numbers less than 4.5; for example, 4, 0, -2

Substitute $a = 4$ in the original inequality.

Left side $= -2.6a + 14.6$	Right side $= -5.2 + 1.8a$
$= -2.6(4) + 14.6$	$= -5.2 + (1.8)(4)$
$= -10.4 + 14.6$	$= -5.2 + 7.2$
$= 4.2$	$= 2$

Since $4.2 > 2$, the left side is greater than the right side, and $a = 4$ satisfies the inequality.

Substitute $a = 0$ in the original inequality.

Left side $= -2.6a + 14.6$	Right side $= -5.2 + 1.8a$
$= -2.6(0) + 14.6$	$= -5.2 + (1.8)(0)$
$= 14.6$	$= -5.2$

Since $14.6 > -5.2$, the left side is greater than the right side, and $a = 0$ satisfies the inequality.

Substitute $a = -2$ in the original inequality.

Left side $= -2.6a + 14.6$	Right side $= -5.2 + (1.8)(-2)$
$= -2.6(-2) + 14.6$	$= -5.2 - 3.6$
$= 5.2 + 14.6$	$= -8.8$
$= 19.8$	

Since $19.8 > -8.8$, the left side is greater than the right side, and $a = -2$ satisfies the inequality.

Since all 3 substitutions verify the inequality, it suggests that $a < 4.5$ is correct.

Example 3 Using an Inequality to Model and Solve a Problem

A super-slide charges \$1.25 to rent a mat and \$0.75 per ride. Haru has \$10.25. How many rides can Haru go on?

a) Choose a variable, then write an inequality to solve this problem.

b) Solve the problem.

c) Graph the solution.

A Solution

a) Let n represent the number of rides that Haru can go on.
The cost of n rides is $1.25 + 0.75n$.
This must be less than or equal to \$10.25.
So, the inequality is:
$1.25 + 0.75n \leq 10.25$

b)

$$1.25 + 0.75n \leq 10.25$$ Subtract 1.25 from each side.

$$1.25 - 1.25 + 0.75n \leq 10.25 - 1.25$$

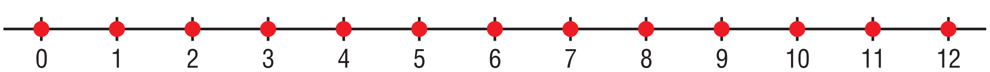
$$0.75n \leq 9$$ Divide each side by 0.75.

$$\frac{0.75n}{0.75} \leq \frac{9}{0.75}$$

$$n \leq 12$$

Haru can go on 12 or fewer rides.

c) Since the number of rides is a whole number, the number line is drawn with a shaded circle at each solution.



Discuss the ideas

1. How is multiplying or dividing each side of an inequality by the same positive number different from multiplying or dividing each side by the same negative number?

2. What is an advantage of substituting 0 for the variable to verify the solution of an inequality? Can you always substitute 0? Explain.

Practice

Check

3. Predict whether the direction of the inequality sign will change when you perform the indicated operation on each side of the inequality.

a) $-9 < -2$; Multiply by 4.

b) $14.5 > 11.5$; Multiply by -3.

c) $6 > -12$; Divide by -4.

d) $-4 < 10$; Divide by 4.

Check your predictions. Were you correct? Explain.

4. Do not solve each inequality. Determine which of the given numbers are solutions of the inequality.

a) $4w < 3$; $-2, 0, 2.5$

b) $3d \geq 5d + 10$; $-5, 0, 5$

5. a) State whether you would reverse the inequality sign to solve each inequality. Then solve and graph the inequality.

i) $10 - y \leq 4$ ii) $3c > -12$

iii) $-6x < 30$ iv) $\frac{m}{-2} < 3$

b) Refer to your solutions in part a. State three values of the variable that satisfy each inequality: one integer, one fraction, and one decimal.

6. A student says that if $c > 9$, then $-3c > -27$. Do you agree? Justify your answer.

Apply

7. Solve each inequality. Verify the solution by substituting 3 different numbers in each inequality.

a) $4 - 2t < 7$

b) $-5x + 2 > 24$

c) $2m + 3 \leq -7$

d) $-4x - 2 > 10$

8. Write, then solve an inequality to show how many cars you would have to wash at \$5 a car to raise at least \$300.

9. Solve each inequality. Graph the solution.

a) $1 - k \leq 4 + k$

b) $2 + 3g < g - 5$

c) $4.5 - 2.5a > 6$

d) $4.7b - 9 \geq 11 - 1.3b$

e) $-6.4 + 3.6s \leq 1.8s + 1.7$

f) $-2.5v + 4.7 \geq -3.8v + 1.58$

10. The Student Council decides to raise money by organizing a dance. The cost of hiring the video-DJ is \$1200 and the Student Council is charging \$7.50 per ticket. How many tickets can be sold to make a profit of more than \$1500?

a) Choose a variable and write an inequality to solve this problem.

b) Use the inequality to solve the problem.

c) Verify the solution and graph it on a number line.

11. Solve each inequality. Graph the solution.

a) $1 + \frac{3}{4}x > 17$

b) $-2 \leq -6 + \frac{1}{4}c$

c) $4 - \frac{2}{3}d \geq \frac{5}{6}d - 5$

d) $\frac{3}{5}f - \frac{1}{2} < 2 + f$

12. Solve each inequality. Show the steps in the solution. Verify the solution by substituting 3 different numbers in each inequality.

a) $4a - 5 \geq a + 2$

b) $15t - 17 \geq 21 - 4t$

c) $10.5z + 16 \leq 12.5z + 12$

d) $7 + \frac{1}{3}b \leq 2b + 22$

13. Jake takes a taxi to tour a city. He is charged \$2.50, plus \$1.20 per kilometre.
Jake has \$12.00. How far can he travel?
a) Choose a variable and write an inequality for this problem.
b) Solve the inequality.
Explain the solution in words.
c) Verify the solution.
d) Graph the solution.

14. Assessment Focus
a) Solve the equation: $2 - \frac{3}{4}w = 3w + \frac{1}{2}$
b) Solve the inequality: $2 - \frac{3}{4}w \geq 3w + \frac{1}{2}$
c) Compare the processes in parts a and b.
How is solving an inequality like solving the related equation? How is it different?
d) Compare the solutions in parts a and b.
How is the solution of an inequality like the solution of the related equation?
How is it different?
Show your work.

15. Janelle plans to replace the light bulbs in her house with energy saver bulbs.
A regular light bulb costs \$0.55 and has an electricity cost of \$0.004 20 per hour.
An energy saver bulb costs \$5.00 and has an electricity cost of \$0.001 05 per hour.
For how many hours of use is it cheaper to use an energy saver bulb than a regular bulb?
a) Write an inequality for this problem.
b) Solve the inequality.
Explain the solution in words.
c) Verify the solution.
d) Graph the solution.

16. Solve each inequality. Graph the solution.
a) $3(0.4h + 5) > 4(0.2h + 7)$
b) $-2(3 - 1.5n) \leq 3(2 - n)$
c) $-4(2.4v - 1.4) \geq -2(0.8 + 1.2v)$
d) $-5(3.2 + 2.3z) < 2(-1.5z - 4.75)$

Take It Further

17. Solve each inequality.
Verify and graph the solution.
a) $\frac{3}{2}a + \frac{1}{2} < \frac{7}{3}a - \frac{3}{4}$
b) $\frac{3}{5}(5.2 - 3m) > -\frac{7}{10}(2m + 7.5)$

18. A business must choose a company to print a promotional brochure.
Company A charges \$900 plus \$0.50 per copy.
Company B charges \$1500 plus \$0.38 per copy.
a) How many brochures must be printed for the cost to be the same at both companies?
b) How many brochures must be printed for Company A to be less expensive?
c) How many brochures must be printed for Company B to be less expensive?
d) Explain the strategies you used to solve these problems.

Reflect

A student says, "Solving inequalities is different from solving equations only when you multiply or divide each side of the inequality by a negative number."
Do you agree with this statement? Explain.

Study Guide

Solving Equations

- An equation is a statement that one quantity is equal to another. To solve an equation means to determine the value of the variable that makes the right side of the equation equal to the left side.

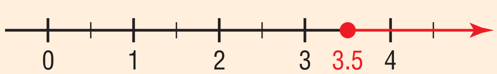
- To solve an equation, isolate the variable on one side of the equation. We use inverse operations or a balance strategy of performing the same operation on both sides of the equation. This can include:
  - adding the same quantity to each side of the equation
  - subtracting the same quantity from each side of the equation
  - multiplying or dividing each side of the equation by the same non-zero quantity

- Algebra tiles, arrow diagrams, and balance scales help model the steps in the solution.

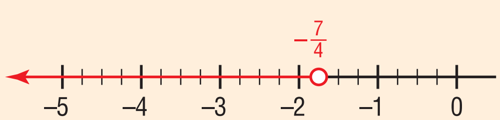
Solving Inequalities

- An inequality is a statement that:
  - one quantity is less than another; for example, $-4 < 3.2a$
  - one quantity is greater than another; for example, $\frac{3}{2}b + 8 > -7$
  - one quantity is greater than or equal to another; for example, $3.4 - 2.8c \geq 1.3c$
  - one quantity is less than or equal to another; for example, $-\frac{5}{8}d + \frac{1}{4} \leq \frac{3}{4} - \frac{1}{2}d$

- The solutions of an inequality are the values of the variable that make the inequality true. We can graph the solutions of an inequality on a number line; for example, $f \geq 3.5$:



and $g < -\frac{7}{4}$:



- The inequality sign reverses when you multiply or divide each side of the inequality by the same negative number.

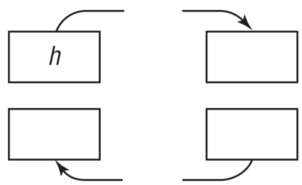
Review

6.1

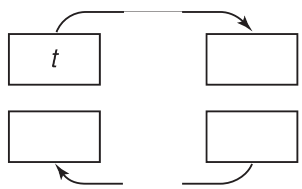
1. a) Copy and complete each arrow diagram to solve each equation.

b) Record the steps in the arrow diagram symbolically.

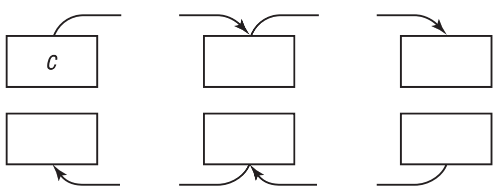
i) $8h = 7.2$



ii) $\frac{t}{5} = -7$



iii) $5c - 1 = 2.4$



2. Both Milan and Daria solve this equation:
$4(3.2s + 5.7) = -6$

a) Milan uses inverse operations to undo the steps used to build the equation.
Show the steps in Milan's solution.

b) Daria uses the distributive property, then inverse operations.
Show the steps in Daria's solution.

c) Describe an advantage and a disadvantage of each method.

3. Solve each equation. Verify the solution.

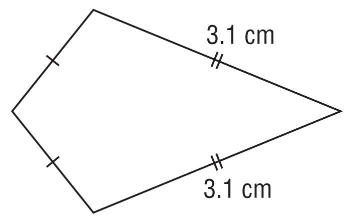
a) $-20.5 = 3b + 16.7$

b) $\frac{t}{3} + 1.2 = -2.2$

c) $-8.5 = 6.3 - \frac{w}{2}$

d) $-2.3(x + 25.5) = -52.9$

4. A kite has longer sides of length 3.1 cm and a perimeter of 8.4 cm.



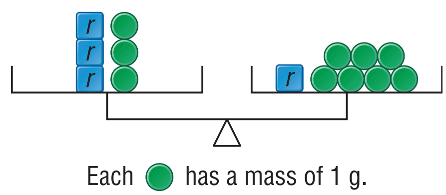
a) Write an equation that can be used to determine the length of a shorter side.

b) Solve the equation.

c) Verify the solution.

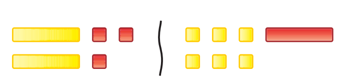
6.2

5. Write the equation represented by these balance scales:



Solve the equation. Record the steps algebraically.

6. Write the equation modelled by these algebra tiles:



Use algebra tiles to solve the equation.
Record the steps algebraically.

7. Solve each equation. Verify the solution.

a) $\frac{-72}{a} = -4.5, a \neq 0$

b) $-\frac{1}{3} + 2m = -\frac{1}{5}$

c) $12.5x = 6.2x + 88$

d) $2.1g - 0.3 = -3.3g - 30$

e) $\frac{3}{2}x + \frac{4}{3} = \frac{5}{8}x + \frac{5}{2}$

f) $5.4(2 - p) = -1.4(p + 2)$

8. Kevin is planning to rent a car for one week. Company A charges \$200 per week, with no charge for the distance driven. For the same car, Company B charges a \$25 administration fee plus \$0.35 per kilometre. Determine the distance driven that will result in equal costs at the two companies.

a) Define a variable and write an equation that can be used to solve the problem.

b) Use the equation to solve the problem.

c) Verify the solution.

9. A student solves this equation:

$$3.5(2v - 5.4) = 2.5(3v - 1.2)$$
$$7v - 5.4 = 7.5v - 1.2$$
$$7v - 7.5v - 5.4 = 7.5v - 7.5v - 1.2$$
$$0.5v - 5.4 + 5.4 = -1.2 + 5.4$$
$$0.5v = 4.2$$
$$\frac{0.5v}{0.5} = \frac{4.2}{0.5}$$
$$v = 8.4$$

What mistakes did the student make? Rewrite a correct and complete solution. How do you know your solution is correct?

6.3 **10.** Define a variable, then write an inequality that describes each situation.

a) Persons under 18 are not admitted.

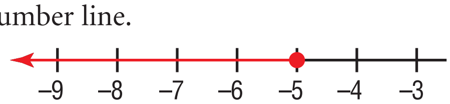
b) A person must be at least 90 cm tall to go on an amusement park ride.

c) Horton can spend a maximum of \$50.

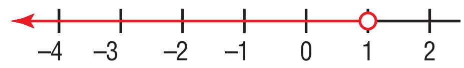
d) A game is recommended for players 5 years and older.

11. Write the inequality represented by each number line.

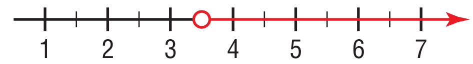
a)



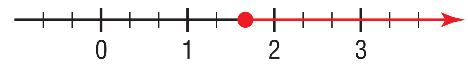
b)



c)



d)



12. a) Graph each inequality on a number line.

i) $a < -5.2$ ii) $b \leq 8.5$

iii) $c > -\frac{5}{3}$ iv) $d \geq \frac{25}{4}$

b) For each inequality in part a, are -3 and 5 possible solutions? Justify your answer.

6.4
6.5 **13.** Determine 3 values of the variable that satisfy each inequality: one integer, one fraction, and one decimal.

a) $h - 2 < -5$ b) $3k > -9$ c) $5 - y > 0$

14. State whether each operation on the inequality $-2x > 5$ will reverse the inequality sign.

a) Multiply each side by 4.

b) Add -5 to each side.

c) Subtract -2 from each side.

d) Divide each side by -6.

15. The cost of a prom is \$400 to rent a hall, and \$30 per person for the meal. The prom committee has \$10 000. How many students can attend?

a) Define a variable and write an inequality to model this problem.

b) Solve the inequality, then graph the solution.

16. Solve each inequality. Verify and graph the solution.

a) $7 + y < 25$ b) $-7y < 14$

c) $\frac{x}{4} > -2.5$ d) $5.2 - y < -5.5$

e) $13.5 + 2y \leq 18.5$ f) $24 + 3a \leq -6 + 7a$

Practice Test

1. Use a model of your choice to illustrate the steps to solve this equation:
$15 + 2d = 5d + 6$
Explain each step and record it algebraically.

2. Solve each equation.

a) $-3x - 0.7 = -7$

b) $\frac{26}{x} = 5 - 1.5$

c) $\frac{r}{3} + 5.4 = -3.2$

d) $2.4w - 5.6 = 3.7 + 1.9w$

e) $\frac{1}{4}c - \frac{7}{2} = \frac{1}{2}c + \frac{3}{4}$

f) $4.5(1.2 - m) = 2.4(-2m + 2.1)$

3. To cater a lunch, Tina's Catering charges \$100, plus \$15 per meal. Norman's Catering charges \$25, plus \$20 per meal. Determine the number of meals that will result in equal costs at the two companies.

a) Define a variable, then write an equation that can be used to solve this problem.

b) Solve the equation. Verify the solution.

4. Solve each inequality. Verify, then graph the solution.

a) $5 - t > 3$

b) $3(t + 2) \geq 11 - 5t$

c) $\frac{m}{4} + 5 \leq \frac{1}{2} - m$

5. A car rental company charges \$24.95 per day plus \$0.35 per kilometre. A business person is allowed \$50 each day for travel expenses. How far can the business person travel without exceeding her daily budget?

a) Define a variable, then write an inequality to solve the problem.

b) Solve the problem. Graph the solution.
How do you know that your answer is correct?

6. Two students wrote these solutions on a test. Identify the errors. Write a correct and complete algebraic solution.

a) $\frac{1}{4}c - 2 = 3$
$4 \times \frac{1}{4}c - 2 = 4 \times 3$
$c - 2 = 12$
$c = 14$

b) $x + 4 < -8 - 2x$
$x + 4 - 4 > -8 - 2x - 4$
$x > -12 - 2x$
$x + 2x > -12 - 2x + 2x$
$3x > 12$
$x > 4$

Unit Problem — Raising Money for the Pep Club

There are 25 students in the school's Pep Club.

1. The Pep Club can buy new uniforms from 2 different suppliers:
Company A charges \$500, plus \$22 per uniform.
Company B charges \$360, plus \$28 per uniform.

a) Define a variable, then write an equation that can be used to determine the number of uniforms that will result in equal costs at both companies.

b) Solve the equation. Verify the solution.

c) Which company should the Pep Club choose? Justify your recommendation.

d) How much money must the Pep Club raise to purchase the uniforms?

2. The Pep Club decides to raise the money for the uniforms by selling snacks at lunch time. The snacks cost the Pep Club \$6.00 for a box of 30.

a) Determine the cost per snack.

b) The Pep Club makes a profit of \$0.25 on each snack sold. Suppose the club does raise the money it needs. Define a variable, then write an inequality that can be used to determine how many snacks might have been sold. How many boxes of snacks did the members of the Pep Club need?

c) Solve the inequality.

d) Verify the solution.

Your work should show:
- an equation and inequality and how you determined them
- how you determined the solutions of the equation and the inequality
- clear explanations of your reasoning

Reflect on Your Learning

How is solving a linear inequality like solving a linear equation?
How is it different?
Include examples in your explanation.

Cumulative Review Units 1–6

1 **1.** Which numbers are perfect squares? Determine the square root of each perfect square. Estimate the square root of each non-perfect square.

a) 3.6 **b)** 0.81 **c)** $\frac{16}{25}$
d) 0.0004 **e)** $\frac{224}{9}$ **f)** 4.41
g) 2.56 **h)** 0.24

2 **2.** Simplify, then evaluate each expression.

a) $(-8)^4 \times (-8)^3 \div (-8)^6$
b) $(9^4 \times 9^3)^0$
c) $[(-2)^5]^3 - [(-3)^3]^2$
d) $[(-4)^1 + (-4)^2 - (-4)^3] \times (-4)^5 \div (-4)^4$
e) $\frac{3^5}{3^2} - (-3)^2$

3 **3.** Evaluate.

a) $1\frac{5}{8} + \left(-4\frac{1}{6}\right)$ **b)** $-3\frac{2}{5} - 7\frac{3}{4}$
c) $(-1.3)(3.4)$ **d)** $\left(-2\frac{1}{10}\right) \div \left(-5\frac{2}{5}\right)$
e) $-8.3 + 6.7 \times (-3.9)$
f) $1\frac{1}{2} \times \left[\left(-\frac{1}{3}\right) + \frac{1}{4}\right]$
g) $[-7.2 - (-9.1)] \div 0.5 + (-0.8)$

4 **4.** The pattern in this table continues.

Term Number, n	Term Value, v
1	5
2	7
3	9
4	11

a) Describe the patterns in the table.
b) Write an equation that relates v to n.
c) Verify the equation by substituting values from the table.
d) Determine the value of the 24th term.
e) Which term number has a value of 233?

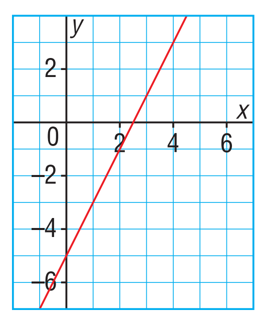
5. a) Create a table of values for the linear relation $y = 3x - 2$.
b) Describe the patterns in the table.
c) Graph the data.

6. a) Does each equation describe a vertical line, a horizontal line, or an oblique line? How do you know?
i) $2x = 5$
ii) $y + 2 = -1$
iii) $x + y = 3$
b) Graph each line in part a. Explain your work.

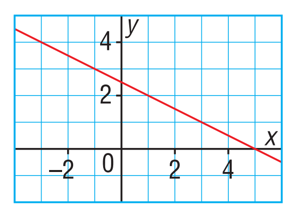
7. Match each equation with a graph below. Justify your answers.

a) $x + 2y = 5$ **b)** $2x + y = 5$ **c)** $2x - y = 5$

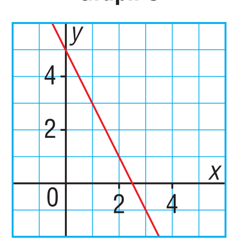
Graph A



Graph B

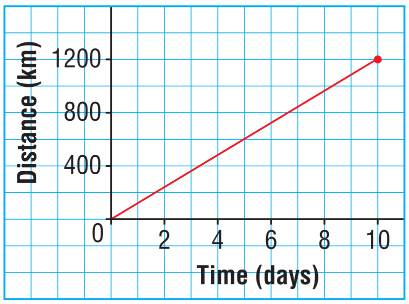


Graph C



8. Carl is cycling across Canada. This graph shows the distance he covers in 10 days.

Carl's Bike Trip



a) Estimate how many days it will take Carl to cycle 700 km.

b) Predict how far Carl will cycle in 13 days.

5 **9.** Name the coefficients, variable, and degree of each polynomial. Identify the constant term if there is one.

a) $3x - 6$ **b)** $4n^2 - 2n + 5$

c) 19 **d)** $-a^2 + 7 - 21a$

10. Simplify each polynomial.

a) $2a - 4 - 9a + 5$

b) $3y - 2y^2 + 4 - y + 3y^2 - 8$

c) $9c - 4cd + d - 6cd + 4 - 7c$

d) $4m^2 - 3n^2 + 2m - 3n + 2m^2 + n^2$

11. Add or subtract the polynomials.

a) $(3s^2 - 2s + 6) + (7s^2 - 4s - 3)$

b) $(8x^2 - 5x + 2) - (5x^2 + 3x - 4)$

c) $(9t - 4 + t^2) + (6 - 2t^2 + 5t)$

d) $(1 + 4n - n^2) - (3n - 2n^2 + 7)$

e) $(6y^2 + 3xy - 2x^2 + 1) + (3x^2 - 2y^2 - 8 + 6xy)$

f) $(8a - 6b - 3a^2 - 2ab) - (4b^2 - 7ab + 9b - 6)$

12. Determine each product or quotient.

a) $9(3s^2 - 7s + 4)$

b) $\dfrac{35 - 49w^2 - 56w}{-7}$

c) $7m(3m - 9)$

d) $(-12d^2 + 18d) \div (-6d)$

6 **13.** Solve each equation. Verify the solution.

a) $9x = 7.2$ **b)** $-2.7 = \frac{a}{4}$

c) $6.5s - 2.7 = -30$ **d)** $\frac{c}{4} - 0.2 = 5.8$

e) $6(n - 8.2) = -18.6$ **f)** $-8 = \frac{7}{c}, c \neq 0$

g) $22 - 7d = -8 - 2d$

h) $3.8v - 17.84 = 4.2v$

i) $2(t - 8) = 4(2t - 19)$

j) $\frac{3}{4}(2r - 4) = \frac{1}{5}(36 - r)$

14. a) Graph each inequality on a number line.

i) $a \leq 3$ **ii)** $-4.5 < b$

iii) $c < -\frac{7}{4}$ **iv)** $d \geq 2\frac{1}{3}$

b) State whether -4 and 2 are possible solutions for each inequality in part a. Justify your answer.

15. Solve each inequality. Graph the solution. Verify the solution.

a) $x + 7 < 3$ **b)** $-3x > 6$

c) $b - 4.8 \geq -1.5$ **d)** $\frac{n}{-8} + 2 \leq -7$

e) $7m + 23 \leq 6m - 15$

f) $6.5 - 0.2t > 8$

g) $-5(4 - 0.8s) \geq 3(19 - s)$

16. Daphne will sell her video game system for \$140 to Surinder. She also offers to sell him video games for \$15 each. Surinder has saved \$210 in total. How many video games can Surinder buy from Daphne?

a) Write an inequality to solve this problem.

b) Solve the inequality. Verify the solution.

UNIT 7

Similarity and Transformations

Here are some flags of different countries and Canadian provinces.
Some of these flags have line symmetry.
Picture each flag lying on your desk.
Which flags have a line of symmetry that is:

- vertical?
- horizontal?
- oblique?

Identify any flag that has more than one line of symmetry.
Which flag has the most lines of symmetry?

What You'll Learn

- Draw and interpret scale diagrams.
- Apply properties of similar polygons.
- Identify and describe line symmetry and rotational symmetry.

Why It's Important

Architects, engineers, designers, and surveyors use similarity and scale diagrams routinely in their work.
Symmetry can be seen in art and nature.
An understanding of symmetry helps us to appreciate and find out more about our world, and to create works of art.

Key Words

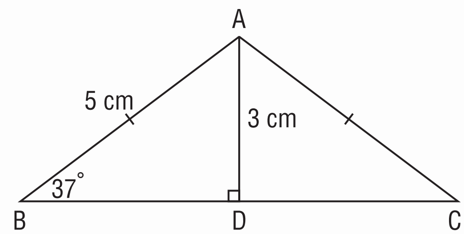
- scale diagram
- corresponding lengths
- scale factor
- proportion
- similar polygons
- corresponding angles
- corresponding sides
- rotational symmetry
- order of rotation
- angle of rotation symmetry

Start Where You Are

What Should I Recall?

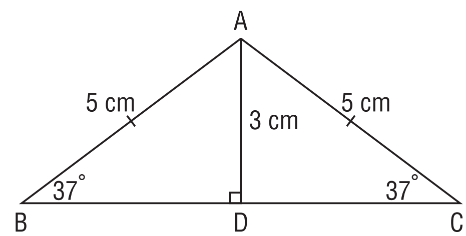
Suppose I have to solve this problem:
Determine the unknown measures of the angles and sides in ΔABC.
The given measures are rounded to the nearest whole number.

I think of what I already know about triangles.
I see that AB and AC have the same hatch marks; this means the sides are equal.
$AC = AB$
So, $AC = 5$ cm

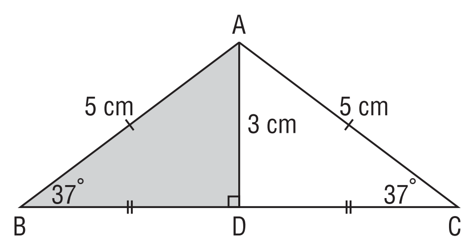


I know that a triangle with 2 equal sides is an isosceles triangle.
So, ΔABC is isosceles.
An isosceles triangle has 2 equal angles that are formed where the equal sides intersect the third side.
I use 3 letters to describe an angle.

$$\begin{aligned} \text{So, } \angle ACD &= \angle ABD \\ &= 37° \end{aligned}$$



Since ΔABC is isosceles, the height AD is the perpendicular bisector of the base BC.
So, $BD = DC$ and $\angle ADB = 90°$
I can use the Pythagorean Theorem in ΔABD to calculate the length of BD.



$$\begin{aligned} AD^2 + BD^2 &= AB^2 \\ 3^2 + BD^2 &= 5^2 \\ 9 + BD^2 &= 25 \\ 9 - 9 + BD^2 &= 25 - 9 \\ BD^2 &= 16 \\ BD &= \sqrt{16} \\ BD &= 4 \end{aligned}$$

BD = 4 cm

So, BC = 2 × 4 cm

= 8 cm

I know that the sum of the angles in a triangle is 180°.

So, I can calculate the measure of ∠BAC.

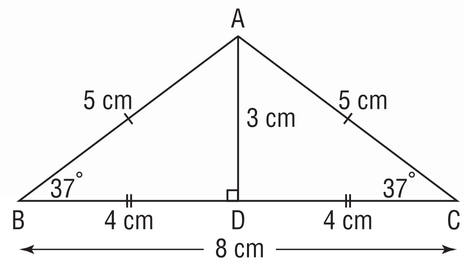
∠BAC + ∠ACD + ∠ABD = 180°

∠BAC + 37° + 37° = 180°

∠BAC + 74° = 180°

∠BAC + 74° − 74° = 180° − 74°

∠BAC = 106°



My friend Janelle showed me a different way to calculate.

She recalled that the line AD is a line of symmetry for an isosceles triangle.

So, ΔABD is congruent to ΔACD.

This means that ∠BAD = ∠CAD

Janelle calculated the measure of ∠BAD in ΔABD.

∠BAD + 37° + 90° = 180°

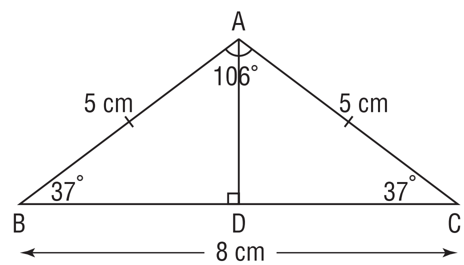
∠BAD + 127° = 180°

∠BAD + 127° − 127° = 180° − 127°

∠BAD = 53°

Then, ∠BAC = 2 × 53°

= 106°



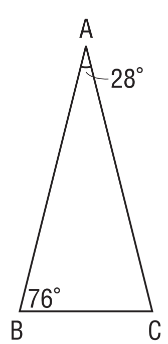
Check

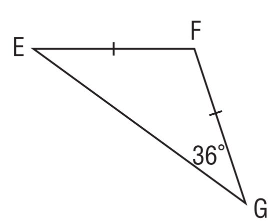
1. Calculate the measure of each angle.

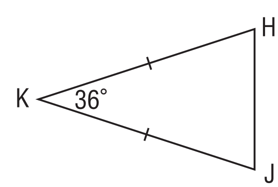
a) ∠ACB

b) ∠GEF and ∠GFE

c) ∠HJK and ∠KHJ







7.1 Scale Diagrams and Enlargements

FOCUS

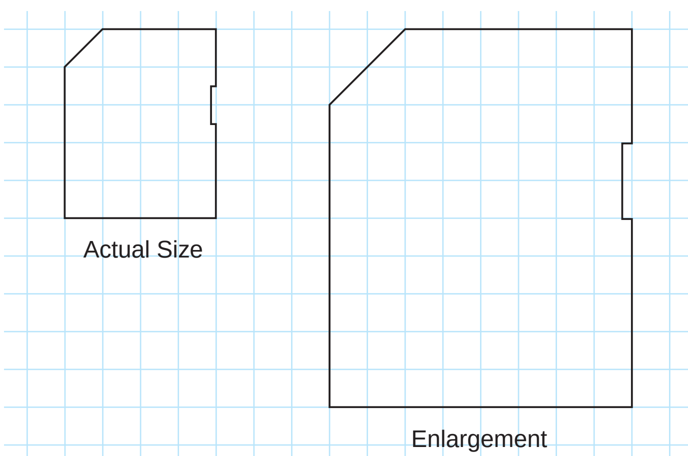
- Draw and interpret scale diagrams that represent enlargements.

How are these photos alike?
How are they different?

Investigate

2

You will need 0.5-cm grid paper.
Here is an actual size drawing of a memory card for a digital camera and an enlargement of the drawing.



➤ Copy the drawings on grid paper.
Measure the lengths of pairs of matching sides on the drawings.
Label each drawing with these measurements.

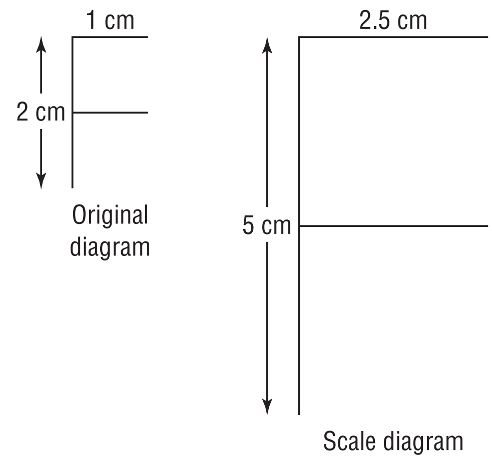
➤ For each measurement, write the fraction: $\frac{\text{Length on enlargement}}{\text{Length on actual size drawing}}$
Write each fraction as a decimal.
What do you notice about these numbers?

Reflect & Share

Compare your numbers with those of another pair of students.
Work together to draw a different enlargement of the memory card.
Determine the fraction $\frac{\text{Length on enlargement}}{\text{Length on actual size drawing}}$ for this new enlargement.

Connect

A diagram that is an enlargement or a reduction of another diagram is called a **scale diagram.**
Here is letter F and a scale diagram of it.



Compare the matching lengths in the scale diagram and in the original diagram.

$$\frac{\text{Length of vertical segment on the scale diagram}}{\text{Length of vertical segment on the original diagram}} = \frac{5 \text{ cm}}{2 \text{ cm}}$$
$$= 2.5$$
$$\frac{\text{Length of horizontal segment on scale diagram}}{\text{Length of horizontal segment on original diagram}} = \frac{2.5 \text{ cm}}{1 \text{ cm}}$$
$$= 2.5$$

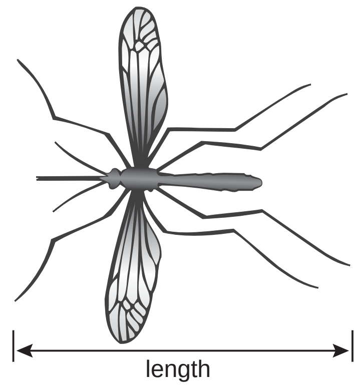
This equation is called a **proportion** because it is a statement that two ratios are equal.

Each length on the original diagram is multiplied by 2.5 to get the matching length on the scale diagram. Matching lengths on the original diagram and the scale diagram are called **corresponding lengths.**

The fraction $\frac{\text{Length on scale diagram}}{\text{Length on original diagram}}$ is called the **scale factor** of the scale diagram.
A scale factor can be expressed as a fraction or as a decimal.
For the diagram above, the scale factor is $\frac{5}{2}$, or 2.5.
Pairs of corresponding lengths have the same scale factor, so we say that corresponding lengths are *proportional.*
Each segment of the enlargement is longer than the corresponding segment on the original diagram, so the scale factor is greater than 1.

Example 1 Using Corresponding Lengths to Determine the Scale Factor

This drawing of a mosquito was printed in a newspaper article about the West Nile Virus. The actual length of the mosquito is 12 mm. Determine the scale factor of the diagram.



A Solution

Measure the length on the scale drawing of the mosquito, to the nearest millimetre.
The length is 4.5 cm, which is 45 mm.

To calculate the scale factor, the units of length must be the same.

The scale factor is: $\frac{\text{Length on scale diagram}}{\text{Length of mosquito}} = \frac{45 \text{ mm}}{12 \text{ mm}}$
$= 3.75$

The scale factor is 3.75.

Example 2 Using a Scale Factor to Determine Dimensions

This photo of longhouses has dimensions 9 cm by 6 cm.
The photo is to be enlarged by a scale factor of $\frac{7}{2}$.
Calculate the dimensions of the enlargement.

Solutions

To determine a length on the scale diagram, multiply the corresponding length on the original diagram by the scale factor.

Method 1

Use mental math.

Length of enlargement: $\frac{7}{2} \times 9 \text{ cm} = \frac{7 \times 9 \text{ cm}}{2}$

$= 31.5 \text{ cm}$

Width of enlargement: $\frac{7}{2} \times 6 \text{ cm} = \frac{7 \times 6 \text{ cm}}{2}$

$= 21 \text{ cm}$

The dimensions of the enlargement are 31.5 cm by 21 cm.

Method 2

Use a calculator.

Write $\frac{7}{2}$ as 3.5.

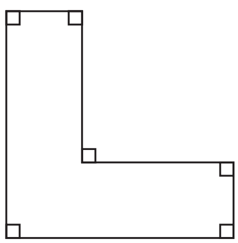
Length of enlargement: $3.5 \times 9 \text{ cm} = 31.5 \text{ cm}$

Width of enlargement: $3.5 \times 6 \text{ cm} = 21 \text{ cm}$

The dimensions of the enlargement are 31.5 cm by 21 cm.

Example 3 Drawing a Scale Diagram that Is an Enlargement

Draw a scale diagram of this metal bracket. Use a scale factor of 1.5.

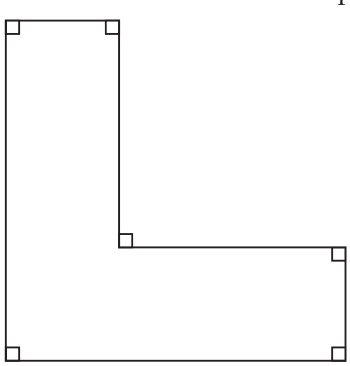


Solutions

Method 1

Use a photocopier. Write the scale factor 1.5 as a percent: 150%

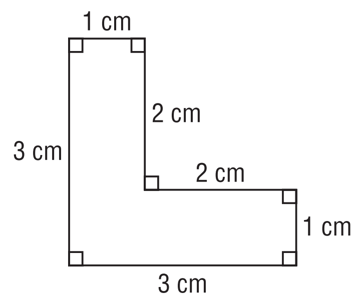
Set the zoom feature on the photocopier to 150%. Copy the diagram.



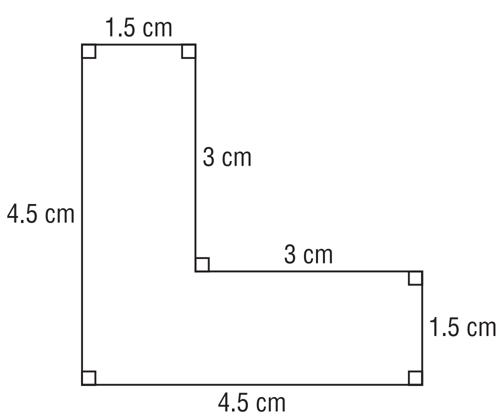
Method 2

Measure the length of each line segment in the given diagram.
Determine the length of each line segment in the scale diagram by multiplying each length on the original diagram by 1.5.

$1.5 \times 3\text{ cm} = 4.5\text{ cm}$
$1.5 \times 2\text{ cm} = 3\text{ cm}$
$1.5 \times 1\text{ cm} = 1.5\text{ cm}$



Use a ruler and a protractor to draw a scale diagram with the new lengths above.
The angles in the scale diagram match the angles in the given diagram.



Discuss the ideas

1. Explain what is meant by the term "scale factor" for a scale diagram.

2. When you calculate a scale factor, why is it important to have the same units for the lengths on the original diagram and the scale diagram?

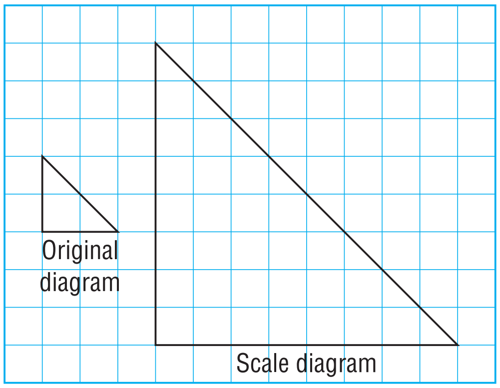
3. Suppose you are given two diagrams. How can you tell if one diagram is a scale drawing of the other diagram?

Practice

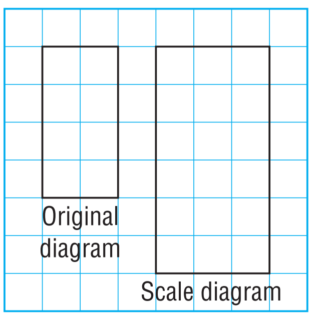
Check

4. Determine the scale factor for each scale diagram.

a)



b)



5. Scale diagrams of different squares are to be drawn. The side length of each original square and the scale factor are given. Determine the side length of each scale diagram.

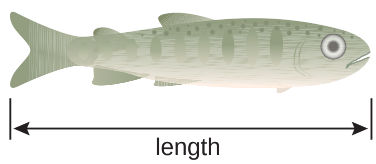
	Side length of original square	Scale factor
a)	12 cm	3
b)	82 mm	$\frac{5}{2}$
c)	1.55 cm	4.2
d)	45 mm	3.8
e)	0.8 cm	12.5

Apply

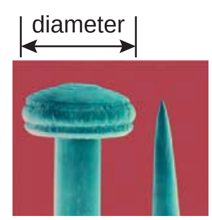
6. A photo of a surfboard has dimensions 17.5 cm by 12.5 cm. Enlargements are to be made with each scale factor below. Determine the dimensions of each enlargement. Round the answers to the nearest centimetre.

a) scale factor 12 **b)** scale factor 20
c) scale factor $\frac{7}{2}$ **d)** scale factor $\frac{17}{4}$

7. Here is a scale diagram of a salmon fry. The actual length of the salmon fry is 30 mm. Measure the length on the diagram to the nearest millimetre. Determine the scale factor for the scale diagram.



8. The head of a pin has diameter 2 mm. Determine the scale factor of this photo of the pinhead.

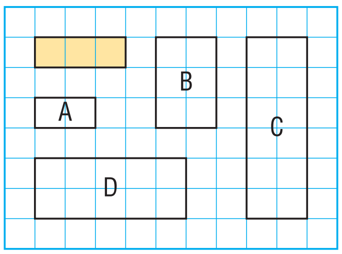


9. This view of the head of a bolt has the shape of a regular hexagon. Each angle is 120°. Use a protractor and ruler to draw a scale diagram of the bolt with scale factor 2.5.

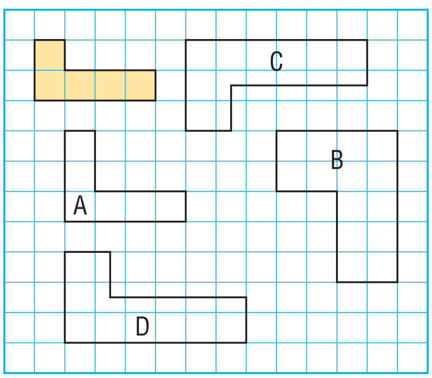
10. Draw your initials on 0.5-cm grid paper. Use different-sized grid paper to draw two different scale diagrams of your initials. For each scale diagram, state the scale factor.

11. **Assessment Focus** For each set of diagrams below, identify which of diagrams A, B, C, and D are scale diagrams of the shaded shape. For each scale diagram you identify:
   - i) State the scale factor.
   - ii) Explain how it is a scale diagram.

a)



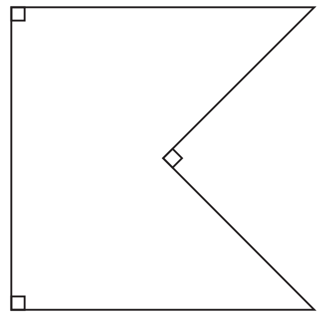
b)



12. One frame of a film in a projector is about 50 mm high. The film is projected onto a giant screen. The image of the film frame is 16 m high.
   a) What is the scale factor of this enlargement?
   b) A penguin is 35 mm high on the film. How high is the penguin on the screen?

13. Look in a newspaper, magazine, or on the Internet. Find an example of a scale diagram that is an enlargement and has its scale factor given. What does the scale factor indicate about the original diagram or object?

14. Draw a scale diagram of the shape below with scale factor 2.5.



15. On a grid, draw ΔOAB with vertices O(0, 0), A(0, 3), and B(4, 0).
   a) Draw a scale diagram of ΔOAB with scale factor 3 and one vertex at C(3, 3). Write the coordinates of the new vertices.
   b) Is there more than one answer for part a? If your answer is no, explain why no other diagrams are possible. If your answer is yes, draw other possible scale diagrams.

Take It Further

16. One micron is one-millionth of a metre, or 1 m = 10^6 microns.
   a) A human hair is about 200 microns wide. How wide is a scale drawing of a human hair with scale factor 400? Give your answer in as many different units as you can.
   b) A computer chip is about 4 microns wide. A scale diagram of a computer chip is 5 cm wide. What is the scale factor?

Reflect

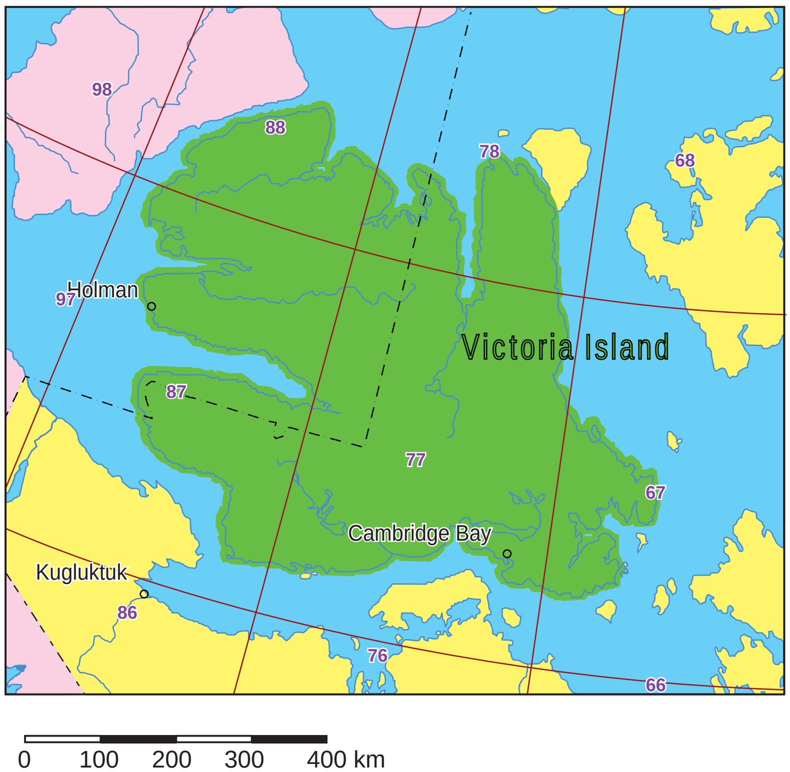
Suppose you are given a scale diagram. Why is it important to know the scale factor?

7.2 Scale Diagrams and Reductions

FOCUS

- Draw and interpret scale diagrams that represent reductions.

Here is a map of Victoria Island from the Internet.
What is the scale on the map? How is the scale used?



Investigate

2

You will need 2-cm grid paper and 0.5-cm grid paper.

➤ Trace your hand on the 2-cm grid paper. Copy this outline of your hand onto the 0.5-cm grid paper. Be as accurate as you can.

➤ On both drawings, measure and label the length of each finger.

For each finger, determine the fraction: $\dfrac{\text{Length on 0.5-cm grid paper}}{\text{Length on 2-cm grid paper}}$

Write each fraction as a decimal to the nearest hundredth.
What do you notice about the decimals?

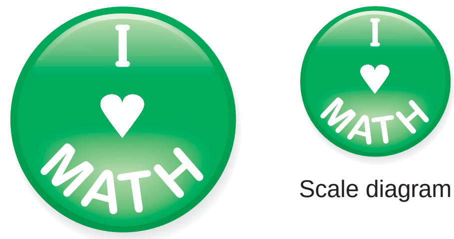
Reflect & Share

Compare your answers with those of another pair of classmates.
Are the numbers the same? Should they be the same? Explain.
How does this work relate to the scale diagrams of the previous lesson?

Connect

A scale diagram can be smaller than the original diagram. This type of scale diagram is called a *reduction*.

Here is a life-size drawing of a button and a scale diagram that is a reduction.



Scale diagram

Original diagram

We measure and compare corresponding lengths in the scale diagram and in the original diagram.

$$\frac{\text{Diameter of scale diagram}}{\text{Diameter of original diagram}} = \frac{2\text{ cm}}{3\text{ cm}} = \frac{2}{3}$$

$$\frac{\text{Height of heart on scale diagram}}{\text{Height of heart on original diagram}} = \frac{0.4\text{ cm}}{0.6\text{ cm}} = \frac{0.4}{0.6} = \frac{2}{3}$$

Write an equivalent fraction.

The fraction $\frac{\text{Length on scale diagram}}{\text{Length on original diagram}}$ is the scale factor of the scale diagram.

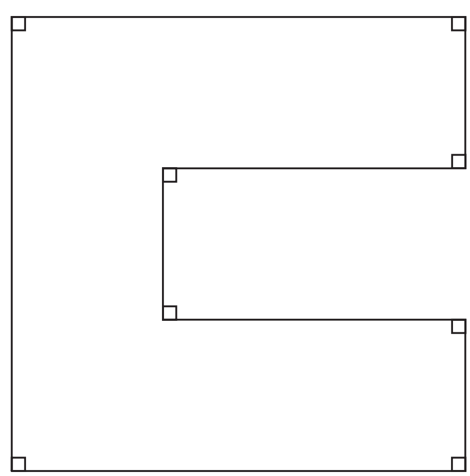
Pairs of corresponding lengths are proportional, and the scale factor is $\frac{2}{3}$.

The equation $\frac{\text{Length on scale diagram}}{\text{Length on original diagram}} = \frac{2}{3}$ is a proportion.

Each side of the reduction is shorter than the corresponding side on the original diagram, so the scale factor is less than 1.

Example 1 Drawing a Scale Diagram that Is a Reduction

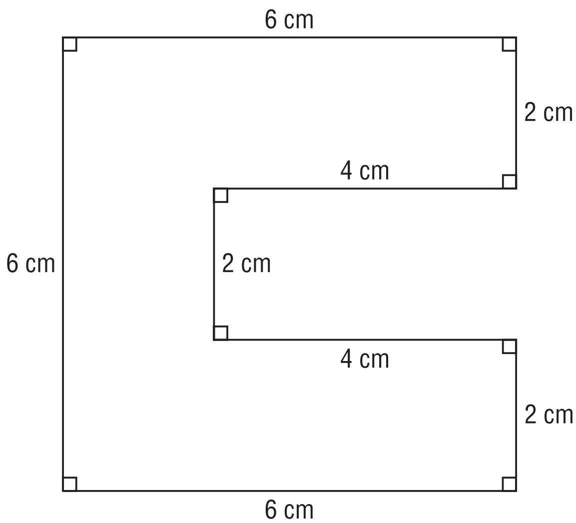
Draw a scale diagram of this octagon.
Use a scale factor of 0.25.



Solutions

Method 1

Measure the length of each line segment in the octagon.



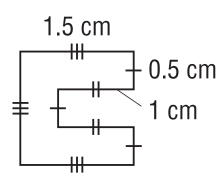
Determine the length of each line segment in the scale diagram by multiplying each length by 0.25.

$0.25 \times 2 \text{ cm} = 0.5 \text{ cm}$

$0.25 \times 4 \text{ cm} = 1 \text{ cm}$

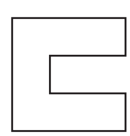
$0.25 \times 6 \text{ cm} = 1.5 \text{ cm}$

Use a ruler and protractor to draw a scale diagram with the new lengths above.
The angles in the scale diagram match the angles in the original diagram.



Method 2

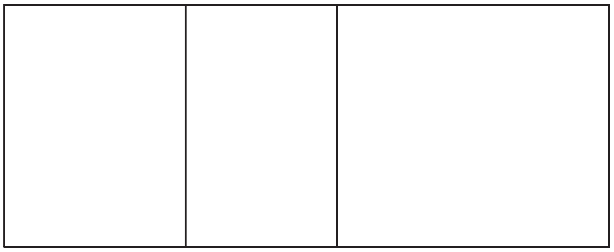
Use a photocopier.
Write the scale factor 0.25 as a percent: 25%
Set the zoom feature on the photocopier to 25%.
Copy the diagram.



A scale may be given as a ratio. For example, suppose the scale on a scale diagram of a house is 1:150. This means that 1 cm on the diagram represents 150 cm, or 1.5 m on the house.

Example 2 Using a Scale on a Scale Diagram to Determine Lengths

Here is a scale diagram of the top view of a truck.



Scale 1:50

The length of the truck is 4 m.

a) The front and back wheels of the truck are 3.85 m apart.
How far apart should the wheels be on the scale diagram?

b) What is the width of the truck?

A Solution

The scale is 1:50. This means that 1 cm on the diagram represents 50 cm on the truck.
So, the scale factor is $\frac{1}{50}$.

a) The front and back wheels of the truck are 3.85 m apart.
Each distance on the scale diagram is $\frac{1}{50}$ of its distance on the truck.
So, on the scale diagram, the distance between the wheels is:

$$\frac{1}{50} \times 3.85 \text{ m} = \frac{3.85 \text{ m}}{50}$$
$$= 0.077 \text{ m}$$

Convert this length to centimetres: $0.077 \text{ m} = 0.077 \times 100$ cm, or 7.7 cm
On the scale diagram, the wheels are 7.7 cm apart.

b) Measure the width of the truck on the scale diagram.
The width is 3.2 cm.
Each actual measure is 50 times as great as the measure on the scale diagram.
So, the actual width of the truck is: $50 \times 3.2 \text{ cm} = 160$ cm
The truck is 160 cm wide; that is 1.6 m wide.

Discuss the ideas

1. What is a reduction? How is it like an enlargement?
How is it different?

2. What is a proportion? When can it be used to solve a problem involving reductions?

3. How can you tell whether a scale diagram is an enlargement or a reduction?

Practice

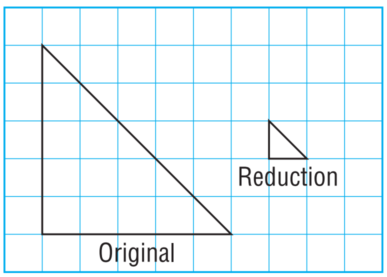
Check

4. Write each fraction in simplest form, then express it as a decimal.

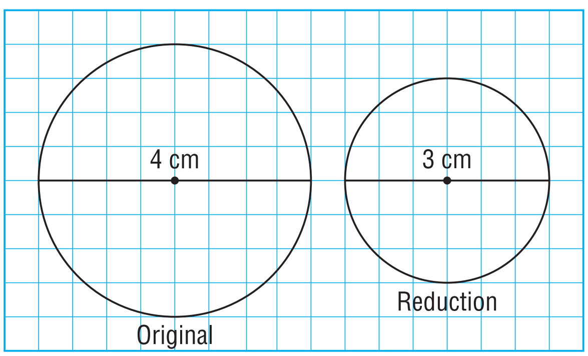
a) $\frac{25}{1000}$ **b)** $\frac{5}{125}$ **c)** $\frac{2}{1000}$ **d)** $\frac{3}{180}$

5. Determine the scale factor for each reduction as a fraction or a decimal.

a)



b)

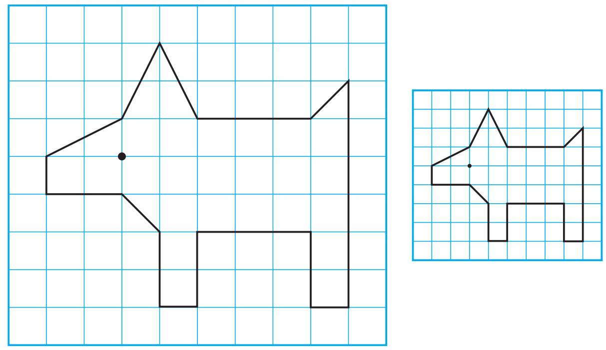


6. For each pair of circles, the original diameter and the diameter of the reduction are given. Determine each scale factor as a fraction or a decimal.

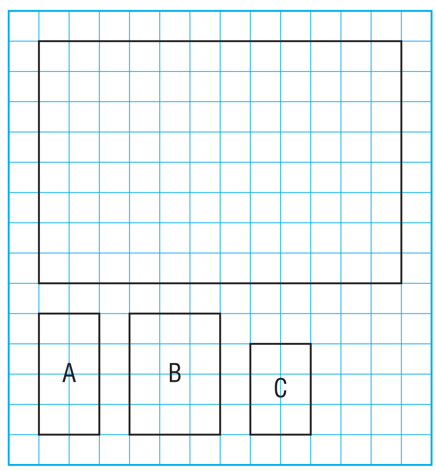
	Diameter of Actual Circle	Diameter of Reduction
a)	50 cm	30 cm
b)	30 cm	20 cm
c)	126 cm	34 cm
d)	5 m	2 cm
e)	4 km	300 m

Apply

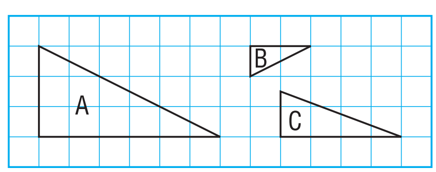
7. Here are two drawings of a dog. Determine the scale factor of the reduction as a fraction and as a decimal.



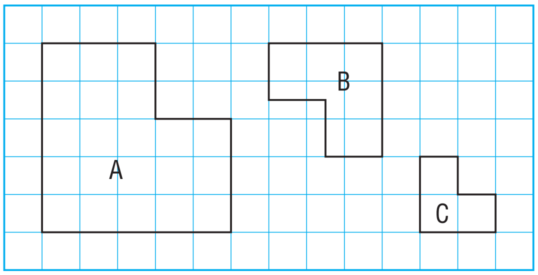
8. Which of rectangles A, B, and C is a reduction of the large rectangle? Justify your answer.



9. Which two polygons have pairs of corresponding lengths that are proportional? Identify the scale factor for the reduction.



10. Which two polygons have pairs of corresponding lengths that are proportional? Identify the scale factor for the reduction.



11. A reduction of each object is to be drawn with the given scale factor. Determine the corresponding length in centimetres on the scale diagram.

a) A desk has length 75 cm.
The scale factor is $\frac{1}{3}$.

b) A bicycle has a wheel with diameter about 60 cm. The scale factor is $\frac{3}{50}$.

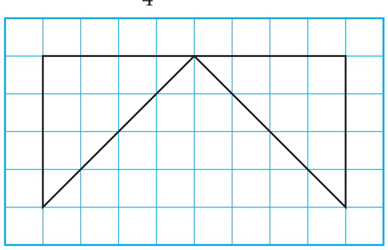
c) A surfboard has length 200 cm.
The scale factor is 0.05.

d) A sailboat has length 8 m.
The scale factor is 0.02.

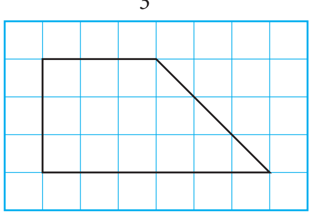
e) A canyon has length 12 km.
The scale factor is 0.000 04.

12. Copy each diagram on 1-cm grid paper. Draw a reduction of each diagram with the given scale factor.

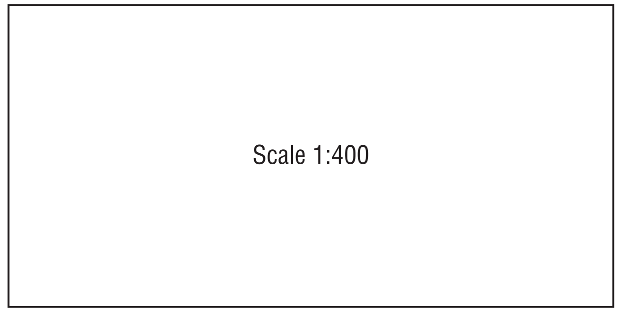
a) scale factor $\frac{3}{4}$



b) scale factor $\frac{2}{3}$



13. Here is a scale diagram of an outdoor hockey rink. The rink is 32 m long.



a) Each hockey net is 1.82 m long. Suppose you had to include the hockey nets on the scale diagram. How long would each hockey net be on the diagram?

b) What is the width of the rink?

14. A volleyball court measures approximately 18 m by 9 m. Make a scale drawing of the court using a scale factor of $\frac{1}{200}$. Show any calculations you made.

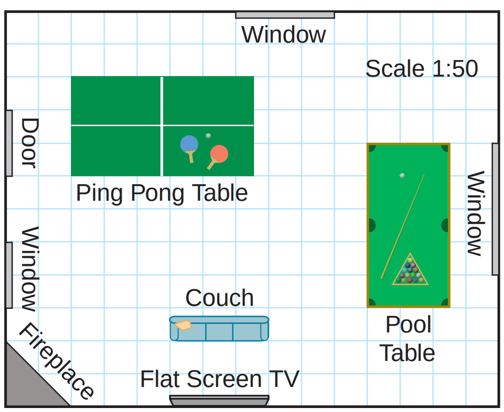
15. A lacrosse field measures 99 m by 54 m. Make a scale drawing of the field using a scale factor of 0.002. Show any calculations you made.

16. Your teacher will give you the dimensions of your classroom. Choose a scale factor and justify its choice. Draw a scale diagram of your classroom. Include as much detail as possible.

17. **Assessment Focus** Draw a scale diagram of any room in your home. Show as much detail as possible by including items in the room. Show any calculations you make and record the scale factor.

18. Look in a newspaper, magazine, or on the Internet. Find an example of a scale diagram that is a reduction and has its scale factor given. What does the scale factor indicate about the original diagram or object?

19. Ask your teacher for a scale diagram of the room shown below. The length of the room is 7.5 m.



a) Determine the scale factor.
b) What are the actual dimensions of:
 i) the ping pong table?
 ii) the pool table?
c) What is the actual size of the flat screen television?
d) Moulding is to be placed around the ceiling. It costs $4.99/m. How much will the moulding cost?

20. A 747 jet airplane is about 70 m long. A plastic model of this plane is 28 cm long.
a) Determine the scale factor of the model.
b) On the model, the wingspan is 24 cm. What is the wingspan on the 747 plane?
c) On the model, the tail is 7.6 cm high. What is the height of the tail on the 747 plane?

Take It Further

21. The approximate diameter of each planet in our solar system is given below.
Earth: 12 760 km; Jupiter: 142 800 km;
Mars: 6790 km; Mercury: 4880 km;
Neptune: 49 500 km; Saturn: 120 600 km;
Uranus: 51 120 km; Venus: 12 100 km
Create a scale drawing that includes all the planets. Justify your choice of scale factor. Label each planet with its actual diameter.

Reflect

A scale factor is the ratio of a length on a scale diagram to the actual length. When you know two of these three values, how can you determine the third value? Include an example in each case.

Drawing Scale Diagrams

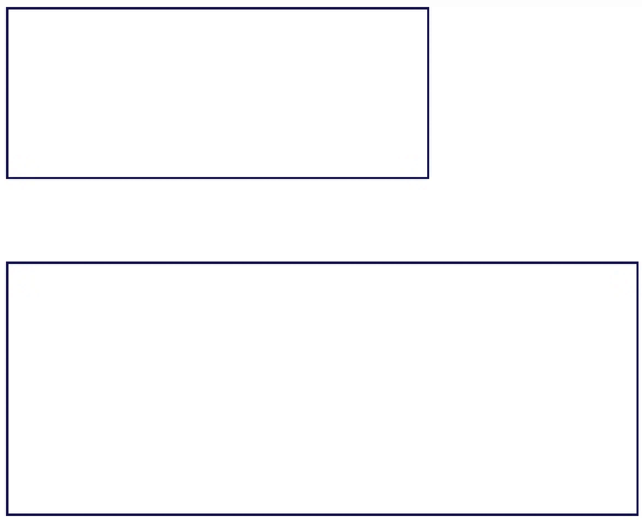
FOCUS

- Use different technologies to produce enlargements and reductions.

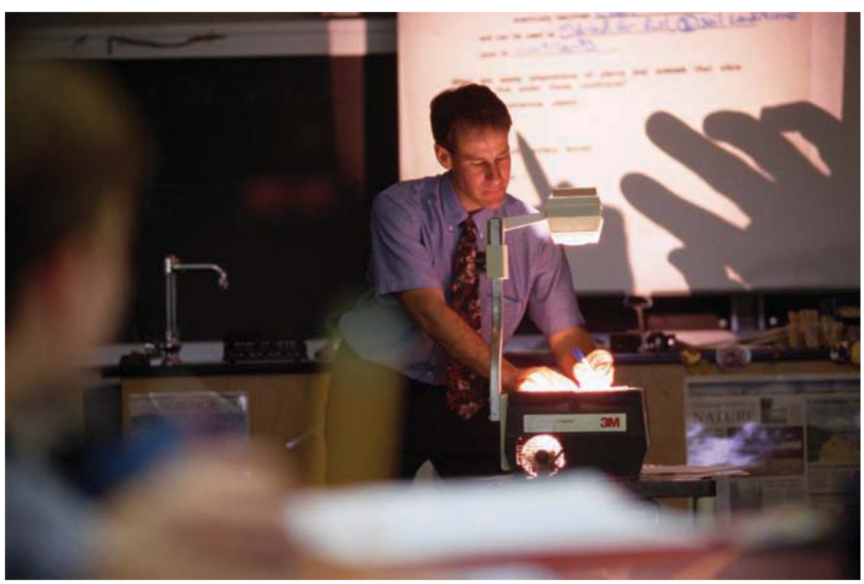
Geometry software can be used to enlarge or reduce a shape.
Use available geometry software.

If you need help at any time, use the software's *Help* menu.

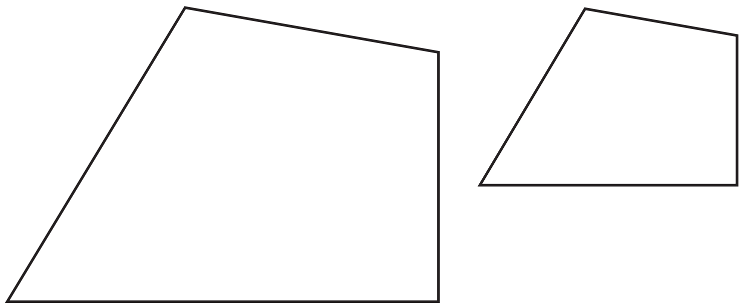
➤ Construct a rectangle. Select the rectangle.
Use the scale feature of the software to enlarge the rectangle.



The rectangle has been enlarged by a scale factor of 1.5, or 150%.



➤ Construct a quadrilateral. Select the quadrilateral.
Use the scale feature to reduce the quadrilateral.



The quadrilateral has been reduced by a scale factor of $\frac{3}{5}$, or 60%.

Check

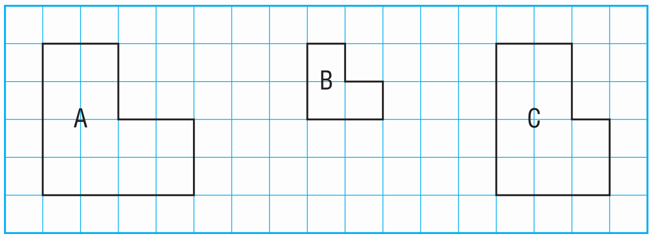
1. Construct a shape. Choose an enlargement scale factor, then enlarge your shape. Calculate the ratios of the corresponding sides of the enlargement and the original shape. What can you say about the ratios?
2. Construct a shape. Choose a reduction scale factor, then reduce your shape. Calculate the ratios of the corresponding sides of the reduction and the original shape. What can you say about the ratios?
3. Print the diagrams of the enlargement and reduction. Trade diagrams with a classmate. Identify the scale factor for each of your classmate's scale diagrams.
4. Try these other ways of enlarging and reducing a shape:
   - an overhead projector
   - a photocopier
   - a Draw tool in a software program

7.3 Similar Polygons

FOCUS

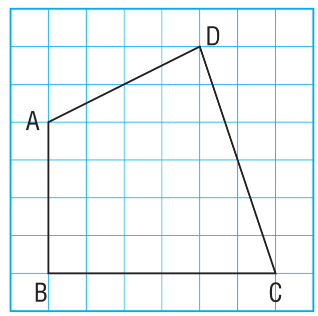
- Recognize and draw similar polygons, then use their properties to solve problems.

Which pair of polygons below show an enlargement or a reduction?
Explain your choice.



Investigate

You will need 0.5-cm grid paper, 2-cm grid paper, a ruler, and a protractor.



➤ Choose a scale factor. Draw an enlargement of quadrilateral ABCD.
Label the new quadrilateral A′B′C′D′.
Measure the side lengths to the nearest millimetre and the angles to the nearest degree.
Copy and complete this table:

Lengths of Sides (mm)	AB	A′B′	BC	B′C′	CD	C′D′	DA	D′A′
Measures of Angles (°)	∠A	∠A′	∠B	∠B′	∠C	∠C′	∠D	∠D′

➤ Choose a scale factor. Draw a reduction of quadrilateral ABCD.
Label the new quadrilateral A″B″C″D″. Copy and complete this table:

Lengths of Sides (mm)	AB	A″B″	BC	B″C″	CD	C″D″	DA	D″A″
Measures of Angles (°)	∠A	∠A″	∠B	∠B″	∠C	∠C″	∠D	∠D″

➤ Copy the table below. Use your results from the first 2 tables to complete this table.
Write the ratios of the lengths of the sides as decimals to the nearest hundredth.

$\frac{AB}{A'B'}$	$\frac{BC}{B'C'}$	$\frac{CD}{C'D'}$	$\frac{DA}{D'A'}$	$\frac{AB}{A''B''}$	$\frac{BC}{B''C''}$	$\frac{CD}{C''D''}$	$\frac{DA}{D''A''}$

➤ What do you notice about the measures of the matching angles?
What do you notice about the ratios of matching sides?

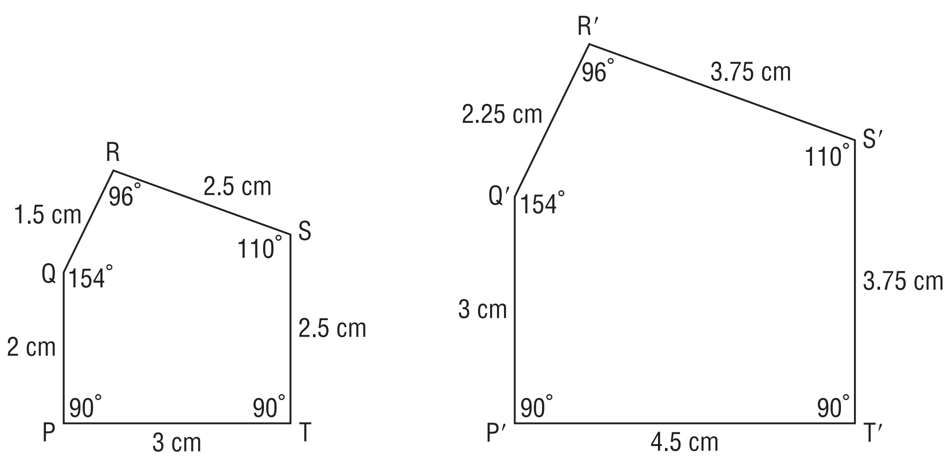
Reflect & Share

Compare your results with those of another pair of students.
Work together to draw two other quadrilaterals that have sides and angles related the same way as your quadrilaterals.
How does this work relate to scale drawings that show enlargements and reductions?

Connect

When one polygon is an enlargement or a reduction of another polygon, we say the polygons are **similar**. Similar polygons have the same shape, but not necessarily the same size.

Here are two similar pentagons.



Matching angles are **corresponding angles**.
Matching sides are **corresponding sides**.
We list the corresponding angles and the pairs of corresponding sides.

Corresponding Sides			Corresponding Angles	
PQ = 2 cm	P′Q′ = 3 cm	$\frac{P'Q'}{PQ} = \frac{3}{2} = 1.5$	∠P = 90°	∠P′ = 90°
QR = 1.5 cm	Q′R′ = 2.25 cm	$\frac{Q'R'}{QR} = \frac{2.25}{1.5} = 1.5$	∠Q = 154°	∠Q′ = 154°
RS = 2.5 cm	R′S′ = 3.75 cm	$\frac{R'S'}{RS} = \frac{3.75}{2.5} = 1.5$	∠R = 96°	∠R′ = 96°
ST = 2.5 cm	S′T′ = 3.75 cm	$\frac{S'T'}{ST} = \frac{3.75}{2.5} = 1.5$	∠S = 110°	∠S′ = 110°
TP = 3 cm	T′P′ = 4.5 cm	$\frac{T'P'}{TP} = \frac{4.5}{3} = 1.5$	∠T = 90°	∠T′ = 90°

In similar polygons:

- pairs of corresponding sides have lengths in the same ratio; that is, the lengths are proportional, and
- the corresponding angles are equal

Pentagon P′Q′R′S′T′ is an enlargement of pentagon PQRST with a scale factor of $\frac{3}{2}$, or 1.5. Or, we can think of pentagon PQRST as a reduction of pentagon P′Q′R′S′T′ with a scale factor of $\frac{2}{3}$.

We say: pentagon PQRST is similar to P′Q′R′S′T′.

We write: pentagon PQRST ~ pentagon P′Q′R′S′T′

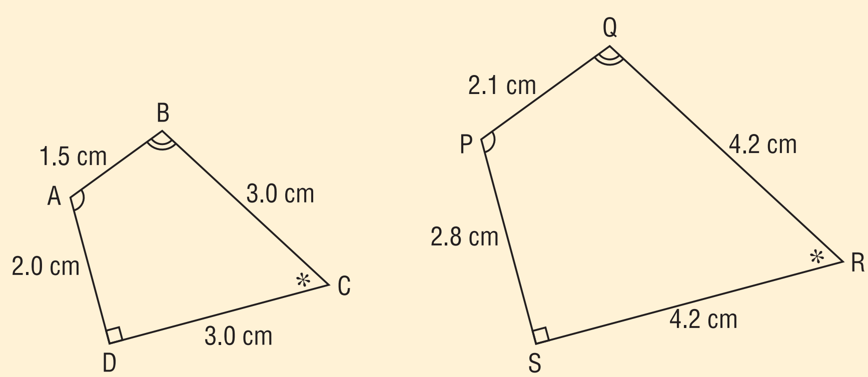
Properties of Similar Polygons

When two polygons are similar:

- their corresponding angles are equal, and
- their corresponding sides are proportional.

It is also true that if two polygons have these properties, then the polygons are similar.

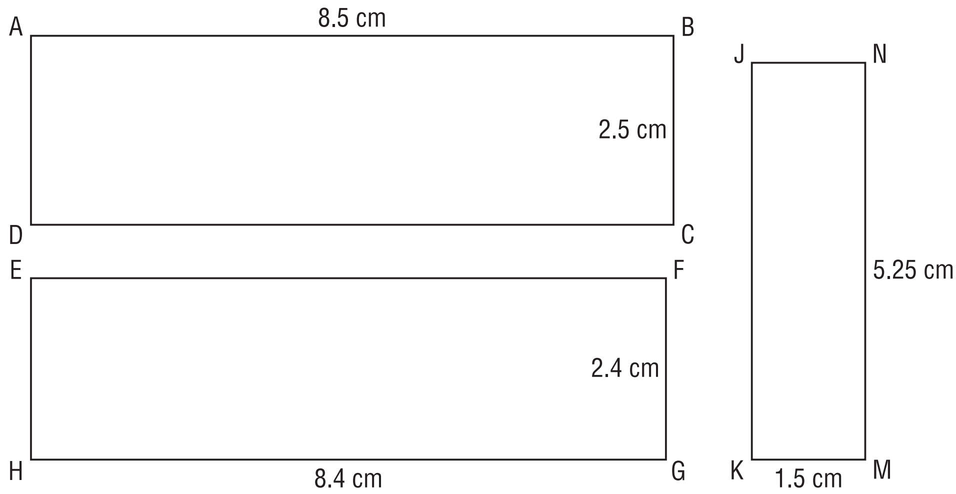
Quadrilateral ABCD ~ quadrilateral PQRS



$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CD}{RS} = \frac{DA}{SP}$$

Example 1 Identifying Similar Polygons

Identify pairs of similar rectangles. Justify the answer.



A Solution

The measure of each angle in a rectangle is 90°.
So, for any two rectangles, their corresponding angles are equal.
For each pair of rectangles, determine the ratios of corresponding sides.
Since the opposite sides of a rectangle are equal, we only need to check the ratios of corresponding lengths and corresponding widths.

For rectangles ABCD and EFGH:

$\frac{AB}{EF} = \frac{8.5}{8.4}$ $\qquad \frac{BC}{FG} = \frac{2.5}{2.4}$

$= 1.011\ldots \qquad = 1.041\overline{6}$

These numbers show that the corresponding sides are not proportional.
So, rectangles ABCD and EFGH are not similar.

For rectangles ABCD and JKMN:

$\frac{AB}{JK} = \frac{8.5}{5.25}$ $\qquad \frac{BC}{KM} = \frac{2.5}{1.5}$

$= 1.619\ldots \qquad = 1.\overline{6}$

These numbers show that the corresponding sides are not proportional.
So, rectangles ABCD and JKMN are not similar.

For rectangles EFGH and JKMN:

$\frac{EF}{JK} = \frac{8.4}{5.25}$ $\qquad \frac{FG}{KM} = \frac{2.4}{1.5}$

$= 1.6 \qquad = 1.6$

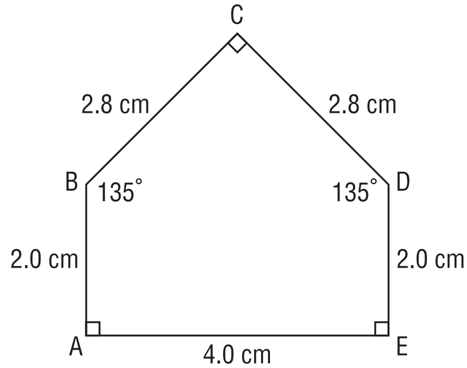
These numbers show that the corresponding sides are proportional.
So, rectangles EFGH and JKMN are similar.

Example 2 Drawing a Polygon Similar to a Given Polygon

a) Draw a larger pentagon that is similar to this pentagon.

b) Draw a smaller pentagon that is similar to this pentagon.

Explain why the pentagons are similar.



A Solution

a) Draw an enlargement. Choose a scale factor greater than 1, such as 2.
Let the similar pentagon be A′B′C′D′E′.
Multiply each side length of ABCDE by 2 to get the corresponding side lengths of A′B′C′D′E′.

$A'B' = 2 \times AB$
$= 2 \times 2.0$ cm
$= 4.0$ cm

$B'C' = 2 \times BC$
$= 2 \times 2.8$ cm
$= 5.6$ cm

$E'A' = 2 \times EA$
$= 2 \times 4.0$ cm
$= 8.0$ cm

Since $DE = AB$,
then $D'E' = A'B'$
$= 4.0$ cm

Since $CD = BC$,
then $C'D' = B'C'$
$= 5.6$ cm

The corresponding angles are equal. So:

$\angle A' = \angle A$
$= 90°$

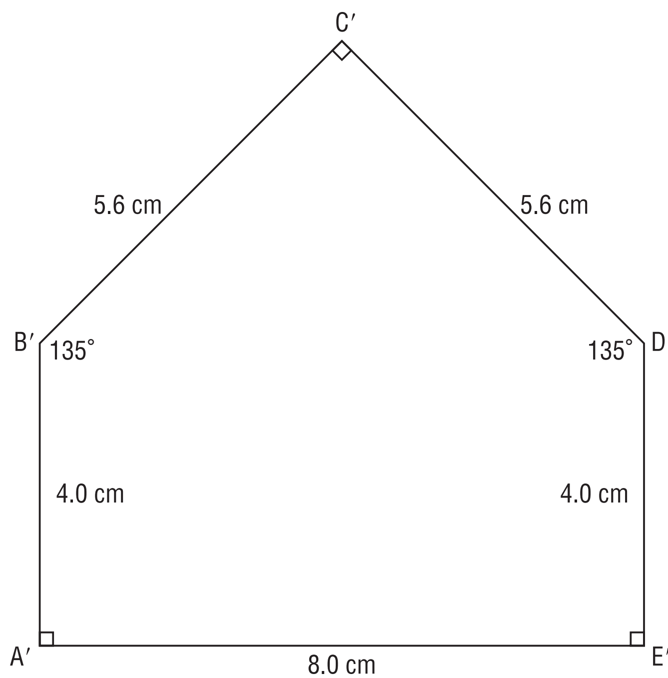
$\angle B' = \angle B$
$= 135°$

$\angle C' = \angle C$
$= 90°$

$\angle D' = \angle D$
$= 135°$

$\angle E' = \angle E$
$= 90°$

Use a ruler and protractor to draw pentagon A′B′C′D′E′.
The pentagons are similar because corresponding angles are equal and corresponding sides are proportional. That is, the length of each side of the enlargement is 2 times the length of the corresponding side of the original pentagon.



b) Draw a reduction. Choose a scale factor that is less than 1, such as $\frac{1}{2}$.
Let the similar pentagon be A′B′C′D′E′.
Multiply each side length of ABCDE by $\frac{1}{2}$ to get the corresponding side lengths of A′B′C′D′E′.

$$A'B' = \frac{1}{2} \times AB = \frac{1}{2} \times 2.0 \text{ cm} = 1.0 \text{ cm}$$

$$B'C' = \frac{1}{2} \times BC = \frac{1}{2} \times 2.8 \text{ cm} = 1.4 \text{ cm}$$

$$E'A' = \frac{1}{2} \times EA = \frac{1}{2} \times 4.0 \text{ cm} = 2.0 \text{ cm}$$

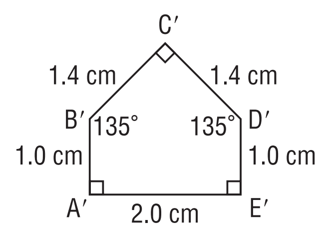
Since DE = AB, then D′E′ = A′B′ = 1.0 cm

Since CD = BC, then C′D′ = B′C′ = 1.4 cm

The corresponding angles are equal. So:

$\angle A' = \angle A = 90°$ $\quad \angle B' = \angle B = 135°$ $\quad \angle C' = \angle C = 90°$ $\quad \angle D' = \angle D = 135°$ $\quad \angle E' = \angle E = 90°$

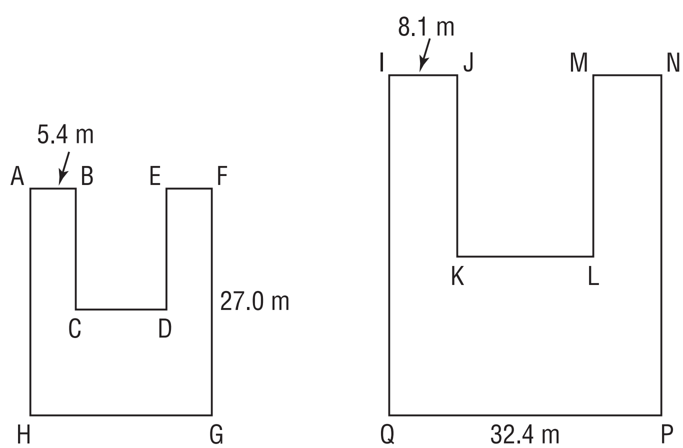
Use a ruler and protractor to draw pentagon A′B′C′D′E′.
The pentagons are similar because corresponding angles are equal and corresponding sides are proportional.
That is, the length of each side of the reduction is $\frac{1}{2}$ the length of the corresponding side of the original pentagon.



Example 3 Solving Problems Using the Properties of Similar Polygons

These two octagonal garden plots are similar.

a) Calculate the length of GH.
b) Calculate the length of NP.



A Solution

a) To calculate GH, consider polygon ABCDEFGH as a reduction of polygon IJKLMNPQ.

The scale factor of the reduction is the ratio of corresponding sides, such as:

$\frac{AB}{IJ} = \frac{5.4}{8.1}$

Write a ratio of corresponding sides that includes GH.

GH corresponds to PQ, so a ratio is $\frac{GH}{PQ}$.

Substitute: PQ = 32.4, then $\frac{GH}{PQ} = \frac{GH}{32.4}$

This ratio is equal to the scale factor.

Use the ratio and scale factor to write a proportion.

$\frac{GH}{32.4} = \frac{5.4}{8.1}$

Solve the proportion to determine GH. Multiply each side by 32.4.

$$32.4 \times \frac{GH}{32.4} = 32.4 \times \frac{5.4}{8.1}$$

$$GH = \frac{32.4 \times 5.4}{8.1}$$

$$GH = 21.6$$

GH is 21.6 m long.

b) To calculate NP, consider polygon IJKLMNPQ as an enlargement of polygon ABCDEFGH. The scale factor of the enlargement is the ratio of corresponding sides, such as: $\frac{IJ}{AB} = \frac{8.1}{5.4}$

Write a ratio of corresponding sides that includes NP.

NP corresponds to FG, so a ratio is $\frac{NP}{FG}$. This ratio is equal to the scale factor.

Substitute: FG = 27.0, then $\frac{NP}{FG} = \frac{NP}{27.0}$

Write a proportion.

$\frac{NP}{27.0} = \frac{8.1}{5.4}$

Solve the proportion to determine NP. Multiply each side by 27.0.

$$27.0 \times \frac{NP}{27.0} = 27.0 \times \frac{8.1}{5.4}$$

$$NP = \frac{27.0 \times 8.1}{5.4}$$

$$= 40.5$$

NP is 40.5 m long.

Discuss the ideas

1. How is drawing a similar polygon like drawing a scale diagram?

2. All rectangles have corresponding angles equal.

a) When would two rectangles be similar?

b) When would two rectangles *not* be similar?

3. How can you tell whether two polygons are similar?

Practice

Check

4. Calculate the side length, in units, in each proportion.

a) $\frac{AB}{8} = \frac{3}{2}$ **b)** $\frac{BC}{25} = \frac{12}{15}$

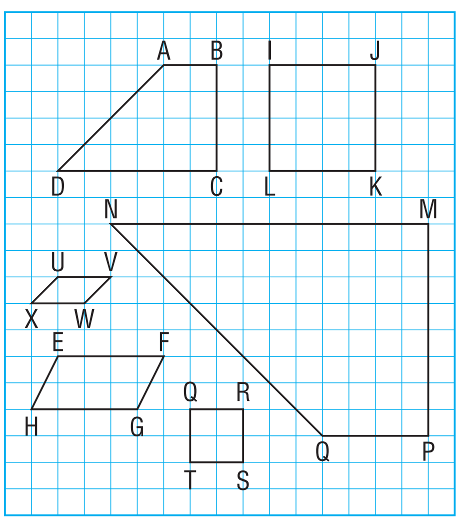
c) $\frac{CD}{4} = \frac{63}{28}$ **d)** $\frac{DE}{7} = \frac{24}{30}$

5. Calculate the value of the variable in each proportion.

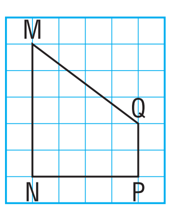
a) $\frac{x}{2.5} = \frac{7.5}{1.5}$ **b)** $\frac{y}{21.4} = \frac{23.7}{15.8}$

c) $\frac{z}{12.5} = \frac{0.8}{1.2}$ **d)** $\frac{a}{0.7} = \frac{1.8}{24}$

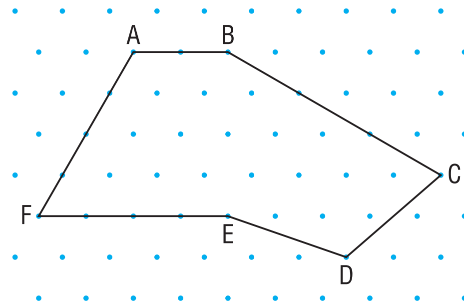
6. Identify similar quadrilaterals. List their corresponding sides and corresponding angles.



7. Use grid paper. Construct a quadrilateral similar to quadrilateral MNPQ.

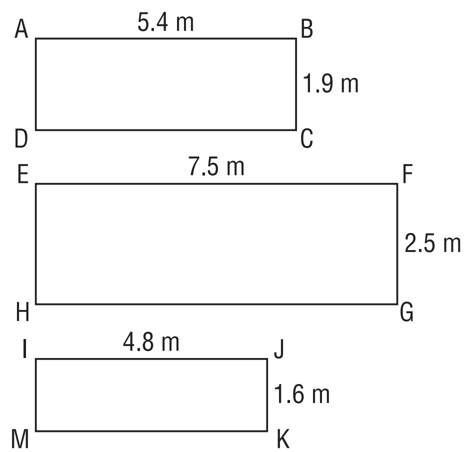


8. Use isometric dot paper. Construct a hexagon similar to hexagon ABCDEF.



Apply

9. Are any of these rectangles similar? Justify your answer.



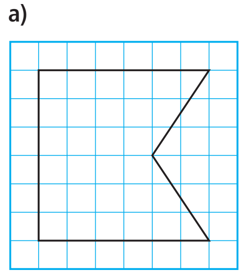
10. For each polygon below:

i) Draw a similar larger polygon.

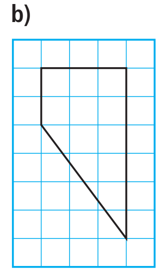
ii) Draw a similar smaller polygon.

Explain how you know the polygons are similar.

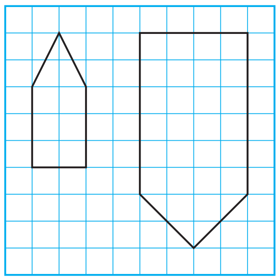
a)

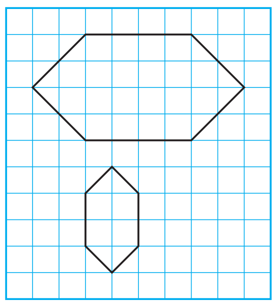


b)



11. Are the polygons in each pair similar? Explain how you know.

a) 

b) 

12. **Assessment Focus** Use grid paper. Construct rectangles with these dimensions: 3 units by 4 units, 6 units by 8 units, 9 units by 12 units, and 12 units by 15 units

a) i) Which rectangle is not similar to the other rectangles?
Explain your reasoning.

ii) Draw two different rectangles that are similar to this rectangle.
Show your work.

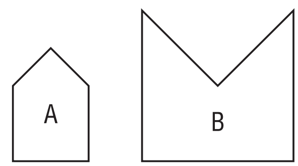
b) The diagonal of the smallest rectangle has length 5 units. Use proportions to calculate the lengths of the diagonals of the other two similar rectangles.

13. A rectangular door has height 200 cm and width 75 cm. It is similar to a door in a doll's house. The height of the doll's house door is 25 cm.

a) Sketch and label both doors.

b) Calculate the width of the doll's house door.

14. Each side of pentagon B is twice as long as a side of pentagon A.



Are the pentagons similar? Explain.

15. Use dot paper.

a) Draw two different:

i) equilateral triangles

ii) squares

iii) regular hexagons

b) Are all regular polygons of the same type similar? Justify your answer.

Take It Further

16. Are all circles similar? Justify your answer.

17. Draw two similar rectangles.

a) What is the ratio of their corresponding sides?

b) What is the ratio of their areas?

c) How are the ratios in parts a and b related?

d) Do you think the relationship in part c is true for all similar shapes?
Justify your answer.

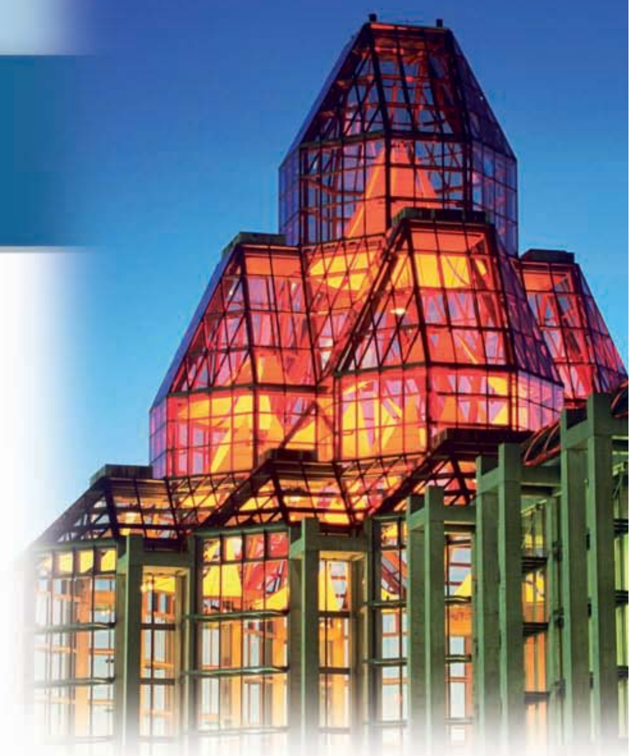
Reflect

What is meant by the statement that two polygons are similar?
How would you check whether two polygons are similar?

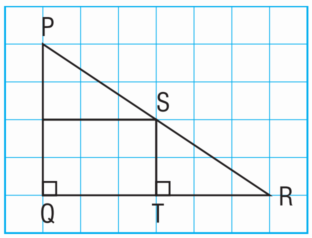
7.4 Similar Triangles

FOCUS

- Use the properties of similar triangles to solve problems.

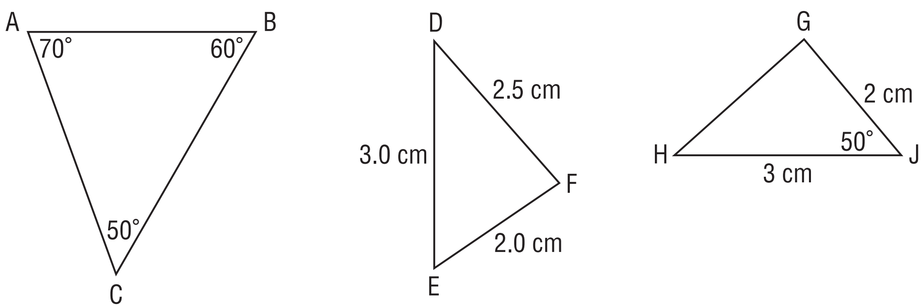


Identify two triangles in this diagram.
How could you find out if they are similar?



Investigate

You will need a ruler, compass, and protractor.
Each pair of students works with one of the three triangles below.



- Construct an enlargement of the triangle you chose. Label its vertices.
  Construct a reduction of the triangle. Label its vertices.
- Measure and record the angles of all your triangles. What do you notice?
- Measure and record the ratios of the lengths of corresponding sides for:
  - the original triangle and its enlargement
  - the original triangle and its reduction

  Write each ratio as a fraction, then as a decimal to the nearest tenth.
  What do you notice about the decimals?
- What can you say about the triangles you worked with?

Reflect & Share

Compare your results with those of another group of classmates.
What do you need to know about two triangles to be able to identify whether they are similar?

Connect

When two polygons are similar:

- the measures of corresponding angles must be equal *and*
- the ratios of the lengths of corresponding sides must be equal.

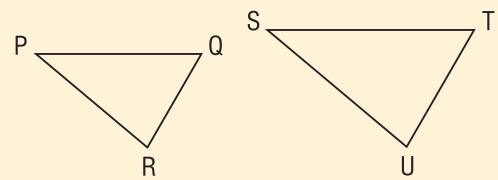
A triangle is a special polygon. When we check whether two triangles are similar:

- the measures of corresponding angles must be equal; *or*
- the ratios of the lengths of corresponding sides must be equal

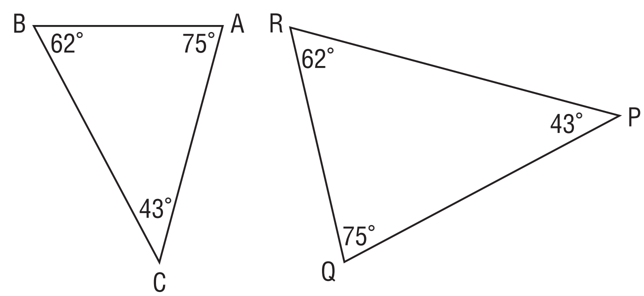
Properties of Similar Triangles

To identify that ΔPQR and ΔSTU are similar, we only need to know that:

- $\angle P = \angle S$ and $\angle Q = \angle T$ and $\angle R = \angle U$; *or*
- $\frac{PQ}{ST} = \frac{QR}{TU} = \frac{PR}{SU}$



These triangles are similar because:
$\angle A = \angle Q = 75°$
$\angle B = \angle R = 62°$
$\angle C = \angle P = 43°$
When we name two similar triangles, we order the letters to match the corresponding angles.
We write: $\Delta ABC \sim \Delta QRP$
Then we can identify corresponding sides:
AB corresponds to QR.
BC corresponds to RP.
AC corresponds to QP.



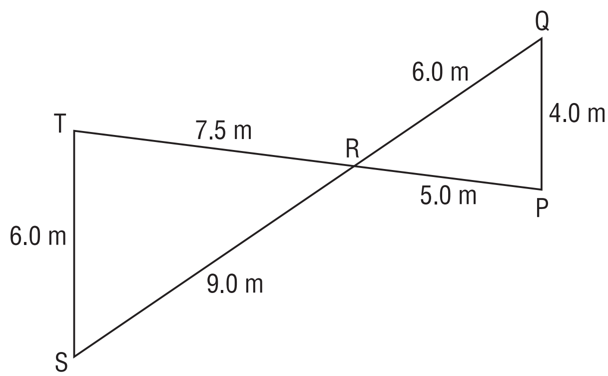
Math Link

Your World

Satellite imagery consists of photographs of Earth taken from space. The images are reductions of regions on Earth. The quality of an image depends upon the instrument used to obtain it and on the altitude of the satellite. The Landsat 7 satellite can create images of objects as small as 10 cm long.

Example 1 Using Corresponding Sides to Name Similar Triangles

Identify the similar triangles.
Justify your answer.



A Solution

Since we know the side lengths of the triangles,
we identify the corresponding sides.
In ΔPQR, from shortest to longest: PQ, PR, QR
In ΔSTR, from shortest to longest: ST, TR, RS
Find out if the corresponding sides are proportional.

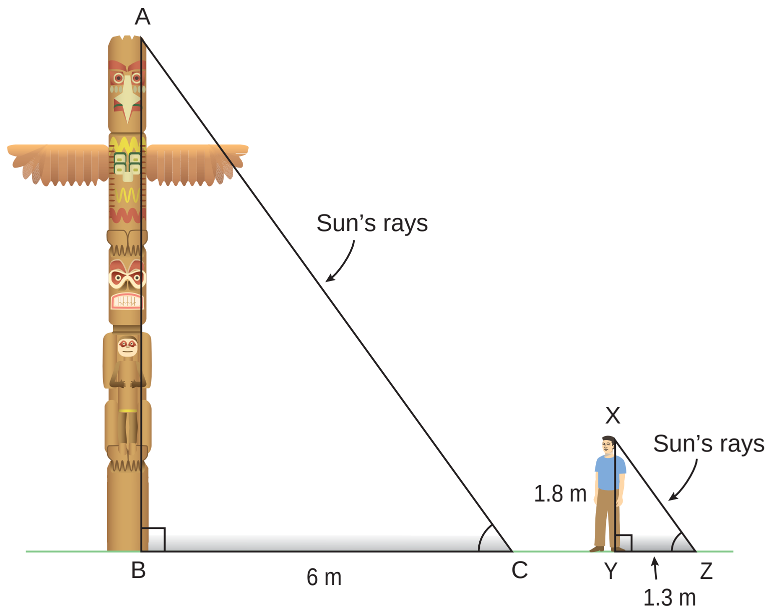
$$\frac{ST}{PQ} = \frac{6.0}{4.0} = 1.5$$
$$\frac{TR}{PR} = \frac{7.5}{5.0} = 1.5$$
$$\frac{RS}{QR} = \frac{9.0}{6.0} = 1.5$$

Since the corresponding sides are proportional, the triangles are similar.
P and T are the vertices where the two shorter sides in each triangle meet,
so $\angle P$ corresponds to $\angle T$.
Similarly, $\angle Q$ corresponds to $\angle S$ and $\angle TRS$ corresponds to $\angle QRP$.
So, $\Delta PQR \sim \Delta TSR$

In *Example 1*, we can say that ΔTSR is an enlargement of ΔPQR
with a scale factor of 1.5.
Or, since $1.5 = \frac{3}{2}$, we can also say that ΔPQR is a reduction of ΔTSR
with a scale factor of $\frac{2}{3}$.
We can use the properties of similar triangles to solve problems
that involve scale diagrams.
These problems involve lengths that cannot be measured directly.

Example 2 Using Similar Triangles to Determine a Length

At a certain time of day, a person who is 1.8 m tall has a shadow 1.3 m long. At the same time, the shadow of a totem pole is 6 m long. The sun's rays intersect the ground at equal angles. How tall is the totem pole, to the nearest tenth of a metre?



A Solution

The sun's rays form two triangles with the totem pole, the person, and their shadows.
If we can show the triangles are similar, we can use a proportion to determine the height of the totem pole.
Assume both the totem pole and the person are perpendicular to the ground, so:
$\angle B = \angle Y = 90°$
The sun's rays make equal angles with the ground, so: $\angle C = \angle Z$
Since two pairs of corresponding angles are equal, the angles in the third pair must also be equal because the sum of the angles in each triangle is 180°.
So, $\angle A = \angle X$
Since 3 pairs of corresponding angles are equal, $\Delta ABC \sim \Delta XYZ$
So, ΔABC is an enlargement of ΔXYZ with a scale factor of $\frac{6}{1.3}$.
Write a proportion that includes the unknown height of the totem pole, AB.

$$\frac{AB}{XY} = \frac{6}{1.3} \qquad \text{Substitute } XY = 1.8.$$

$$\frac{AB}{1.8} = \frac{6}{1.3} \qquad \text{To solve for AB, multiply each side by 1.8.}$$

$$1.8 \times \frac{AB}{1.8} = \frac{6}{1.3} \times 1.8$$

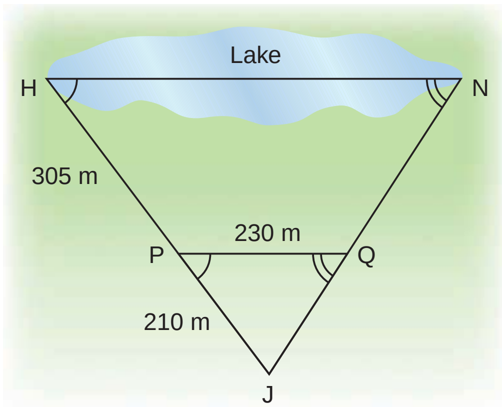
$$AB = \frac{6 \times 1.8}{1.3}$$

$$\doteq 8.308$$

The height of the totem pole is about 8.3 m.

Example 3 Using Overlapping Similar Triangles to Determine a Length

A surveyor wants to determine the width of a lake at two points on opposite sides of the lake. She measures distances and angles on land, then sketches this diagram. How can the surveyor determine the length HN to the nearest metre?



A Solution

Identify the two triangles, then draw them separately.

Consider ΔHNJ and ΔPQJ. From the diagram:

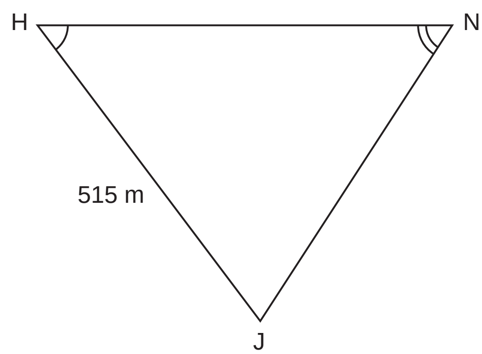
$\angle$NHJ = $\angle$QPJ

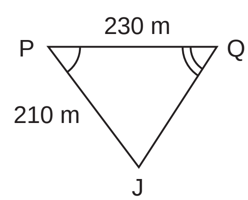
$\angle$HNJ = $\angle$PQJ

$\angle$J is the common angle to both triangles.

Since 3 pairs of corresponding angles are equal,

$\Delta HNJ \sim \Delta PQJ$





Two corresponding sides are:

HJ = 305 m + 210 m   and   PJ = 210 m

= 515 m

So, ΔHNJ is an enlargement of ΔPQJ with a scale factor of $\frac{515}{210}$.

Write a proportion that includes the unknown length HN.

$$\frac{HN}{PQ} = \frac{515}{210}$$ Substitute PQ = 230.

$$\frac{HN}{230} = \frac{515}{210}$$ To solve for HN, multiply each side by 230.

$$230 \times \frac{HN}{230} = \frac{515}{210} \times 230$$

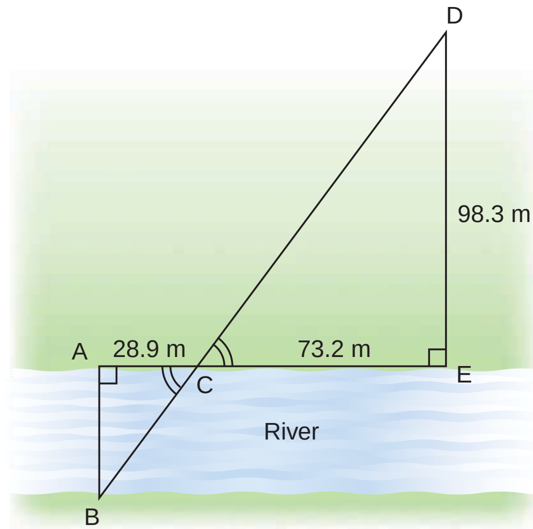
$$HN = \frac{515 \times 230}{210}$$

$$\doteq 564.0476$$

The width of the lake, HN, is about 564 m.

Example 4 Using Triangles Meeting at a Vertex to Determine a Length

A surveyor used this scale diagram to determine the width of a river. The measurements he made and the equal angles are shown. What is the width, AB, to the nearest tenth of a metre?



A Solution

Consider ΔABC and ΔEDC.
From the diagram:
$\angle A = \angle E$
$\angle B = \angle D$
$\angle ACB = \angle ECD$
Since 3 pairs of corresponding angles are equal, $\Delta ABC \sim \Delta EDC$
Two corresponding sides are:
AC = 28.9 m and EC = 73.2 m
So, ΔABC is a reduction of ΔEDC with a scale factor of $\frac{28.9}{73.2}$.

Write a proportion that includes the unknown length AB.

$$\frac{AB}{ED} = \frac{28.9}{73.2}$$ Substitute ED = 98.3.

$$\frac{AB}{98.3} = \frac{28.9}{73.2}$$ To solve for AB, multiply each side by 98.3.

$$98.3 \times \frac{AB}{98.3} = \frac{28.9}{73.2} \times 98.3$$

$$AB = \frac{28.9 \times 98.3}{73.2}$$

$$\doteq 38.8097$$

The width of the river, AB, is about 38.8 m.

Discuss the ideas

1. How can you tell that two triangles are similar?

2. When two triangles are similar, how do you identify the corresponding sides?

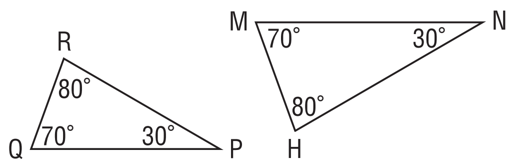
3. Suppose you know that two triangles are similar. How do you write the proportion to determine the length of an unknown side?

Practice

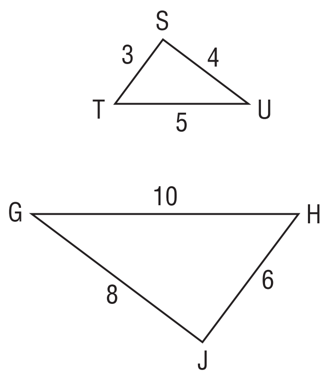
Check

4. Which triangles in each pair are similar? How do you know?

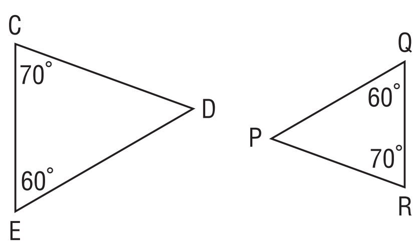
a)



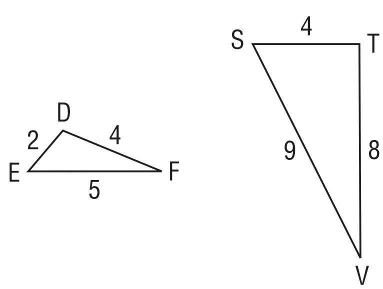
b)



c)

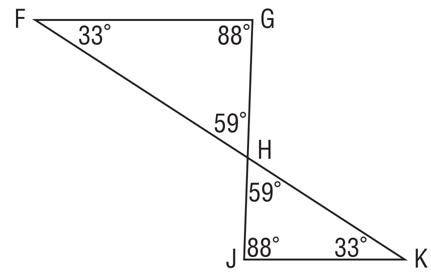


d)

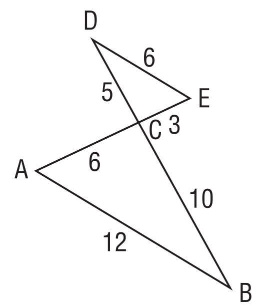


5. In each diagram, identify two similar triangles. Explain why they are similar.

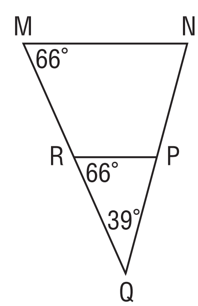
a)



b)



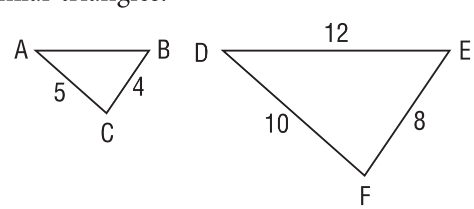
c)



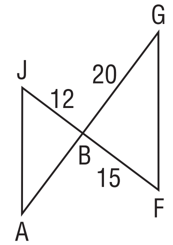
Apply

6. Determine the length of AB in each pair of similar triangles.

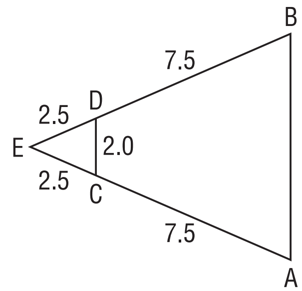
a)



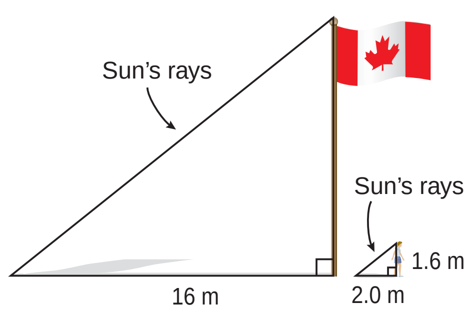
b)



c)



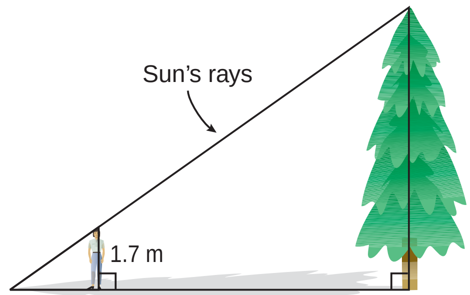
7. Jaquie is 1.6 m tall. When her shadow is 2.0 m long, the shadow of the school's flagpole is 16 m long. How tall is the flagpole, to the nearest tenth of a metre?



8. **Assessment Focus** Work with a partner. Use the method described in question 7. Choose an object whose height you cannot measure directly.

a) Draw a labelled diagram.
b) Indicate which triangles are similar.
c) Determine the height of the object.
Show your work.

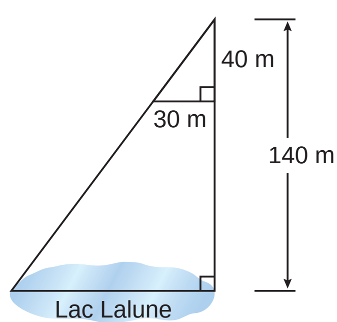
9. Tina wants to estimate the heights of two trees. For each tree, she stands so that one end of her shadow coincides with one end of the shadow of the tree. Tina's friend measures the lengths of her shadow and the tree's shadow. Tina is 1.7 m tall.



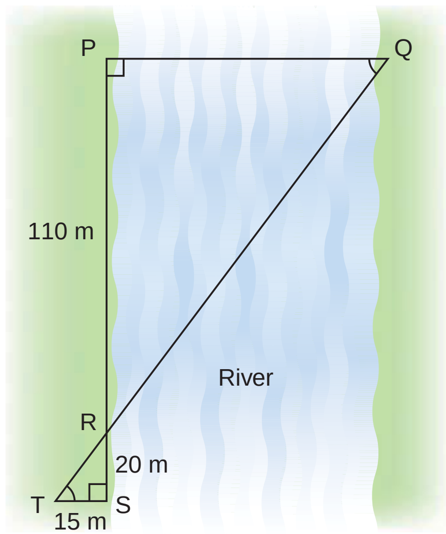
a) Tina's shadow is 2.4 m and the first tree's shadow is 10.8 m. What is the height of the tree?
b) Tina's shadow is 0.8 m and the second tree's shadow is 12.8 m. What is the height of the tree?

10. When the shadow of a building is 16 m long, a 4-m fence post casts a shadow 3 m long.
a) Sketch a diagram.
b) How tall is the building?

11. This scale diagram shows the measurements a surveyor made to determine the length of Lac Lalune. What is this length?
How do you know?

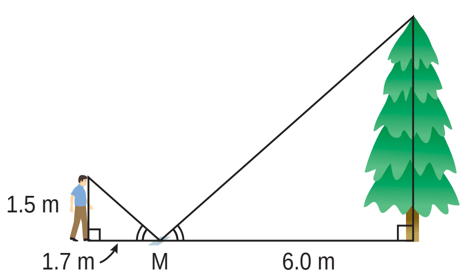


12. To help calculate the distance PQ across a river, Emil drew the diagram below based on measurements he made. What is the distance across the river?

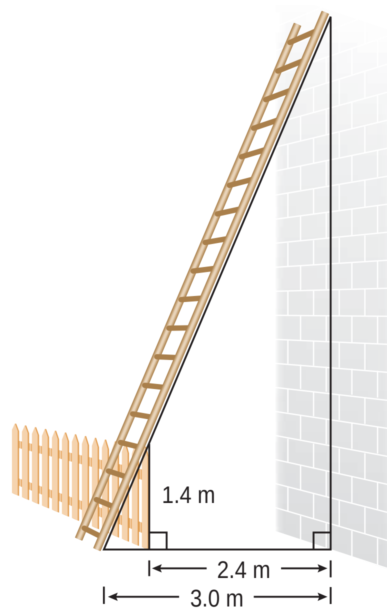


Take It Further

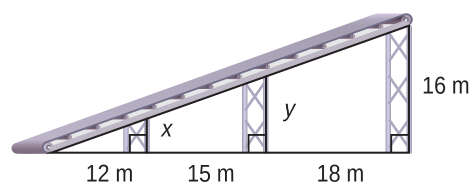
13. Phillipe places a mirror M on the ground 6.0 m from a tree. When he is 1.7 m from the mirror, he can see the top of the tree in the mirror. His eyes are 1.5 m above the ground. The diagram below shows the equal angles. How can you use similar triangles to determine the height of the tree to the nearest tenth of a metre?



14. The foot of a ladder is 3 m from the base of a wall. The ladder just touches the top of a 1.4-m fence that is 2.4 m from the wall. How high up the wall does the ladder reach? How do you know?



15. In the diagram below, how high are the two supports x and y for the conveyor belt?



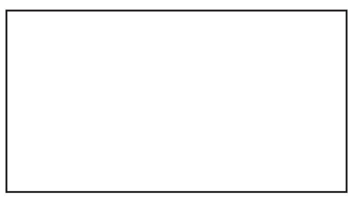
Reflect

How do the properties of similar triangles help you to determine distances that cannot be measured directly? Include an example in your explanation.

Mid-Unit Review

7.1 **1.** A photo of a gymnast is to be enlarged. The dimensions of the photo are 15 cm by 10 cm. What are the dimensions of the enlargement with a scale factor of $\frac{7}{5}$?

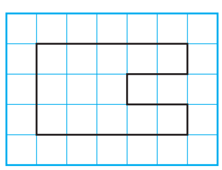
2. A computer chip has dimensions 15 mm by 8 mm. Here is a scale drawing of the chip.



a) Determine the scale factor of the diagram.

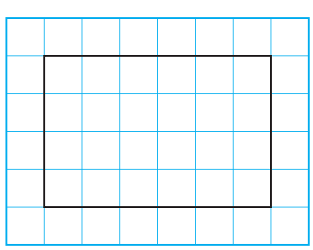
b) Draw a scale diagram of the chip with a scale factor of 8.

7.2 **3. a)** Copy this polygon on 1-cm grid paper.

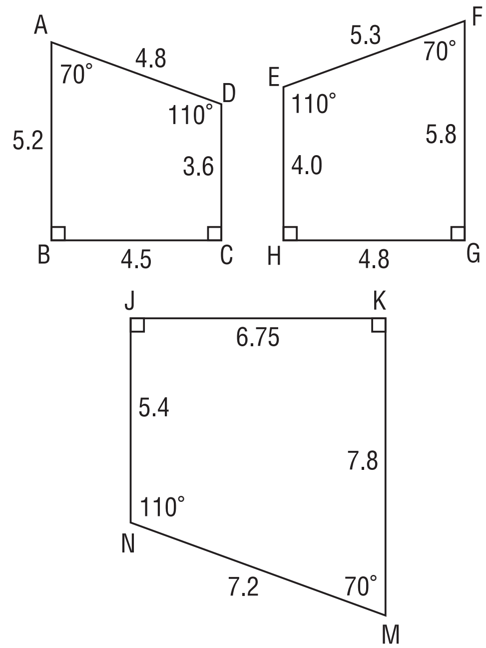


b) Draw a scale diagram of the polygon with a scale factor of $\frac{3}{5}$. Show any calculations you made.

4. This top view of a swimming pool is drawn on 0.5-cm grid paper. The dimensions of the pool are 60 m by 40 m. Determine the scale factor of the reduction as a fraction or a decimal.



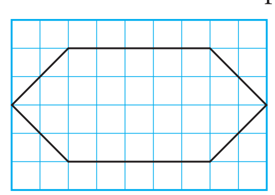
7.3 **5.** These quadrilaterals have corresponding angles equal.



a) Are any of these quadrilaterals similar? Justify your answer.

b) Choose one quadrilateral. Draw a similar quadrilateral. How do you know the quadrilaterals are similar?

6. A window has the shape of a hexagon.



Draw a hexagon that is similar to this hexagon. Explain how you know the hexagons are similar.

7.4 **7.** A tree casts a shadow 8 m long. At the same time a 2-m wall casts a shadow 1.6 m long.

a) Sketch a diagram.

b) What is the height of the tree?

7.5 Reflections and Line Symmetry

FOCUS

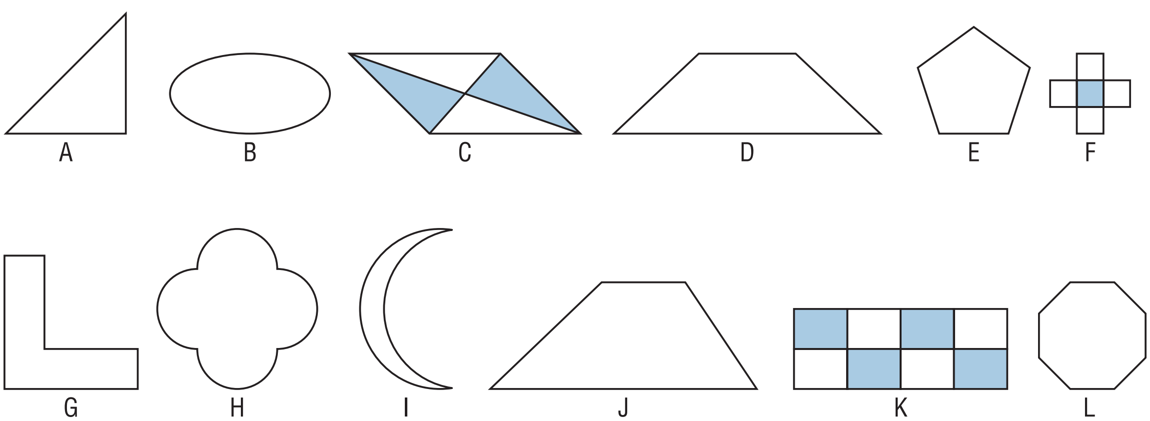
- Draw and classify shapes with line symmetry.

How can you use this photograph to show what you know about line symmetry?

Investigate

2

Your teacher will give you a large copy of the shapes below.



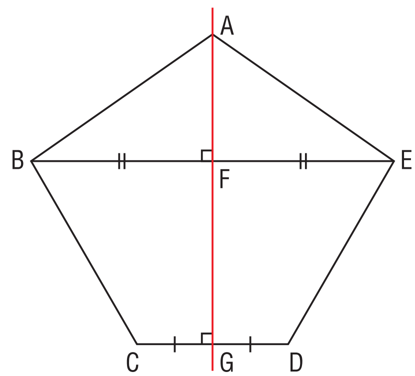
Which shapes have the same number of lines of symmetry?
Sort the shapes according to the number of lines of symmetry they have.
Which shapes do not have line symmetry? How can you tell?

Reflect & Share

Share your sorting with another pair of students.
Compare strategies for identifying the lines of symmetry.

Connect

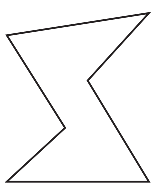
The pentagon ABCDE has one line of symmetry AG, because AG divides the pentagon ABCDE into two congruent parts:
polygon ABCG is congruent to polygon AEDG.



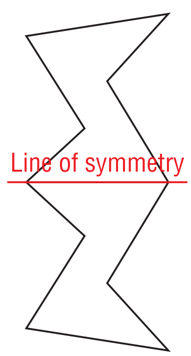
Also, each point on one side of the line of symmetry has a corresponding point on the other side of the line. These two points are the same distance, or *equidistant* from the line of symmetry:
points B and E correspond, BF = FE, and BE ⊥ AG.

A line of symmetry is also called a *line of reflection*. If a mirror is placed along one side of a shape, the reflection image and the original shape together form one larger shape. The line of reflection is a line of symmetry of this larger shape.

Original shape



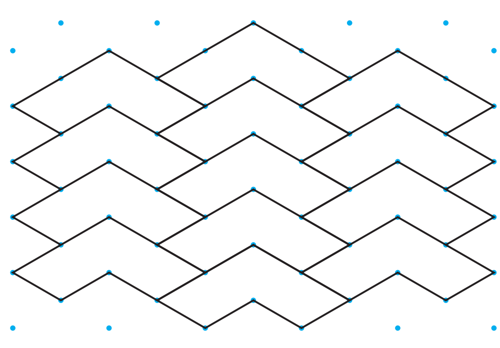
Original shape and its reflection image



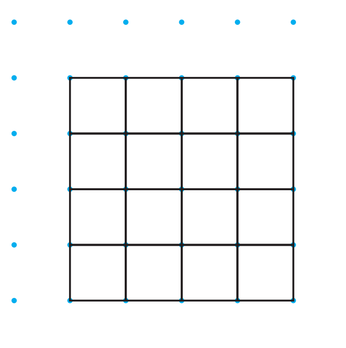
Example 1 Identifying Lines of Symmetry in Tessellations

Identify the lines of symmetry in each tessellation.

a)

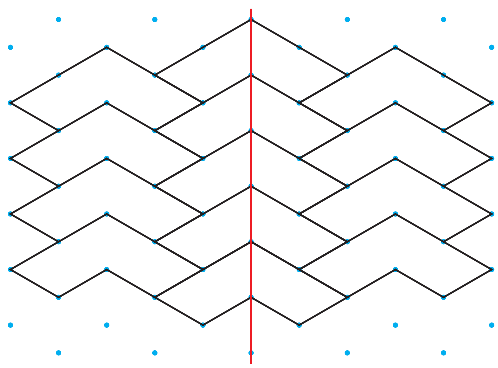


b)

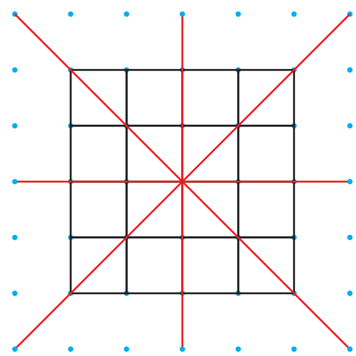


A Solution

a) The red line is the line of symmetry for this tessellation. Each point on one side of the line has a corresponding point on the other side. The pattern on one side of the line of symmetry is a mirror image of the pattern on the other side.



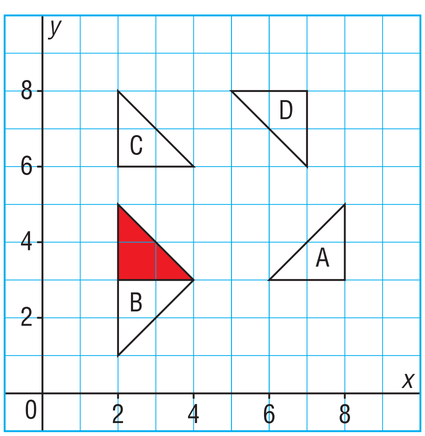
b) This tessellation has 4 lines of symmetry. For each line, a point on one side of the line has a matching point on the other side. And, the pattern on one side of the line is a mirror image of the pattern on the other side.



Two shapes may be related by a line of reflection.

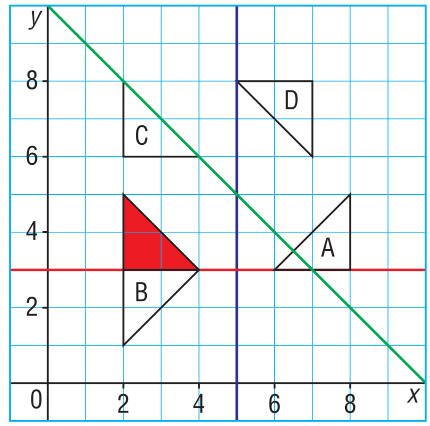
Example 2 Identifying Shapes Related by a Line of Reflection

Identify the triangles that are related to the red triangle by a line of reflection. Describe the position of each line of symmetry.



A Solution

Triangle A is the reflection image of the red triangle in the blue line through 5 on the x-axis.
Triangle B is the reflection image of the red triangle in the red line through 3 on the y-axis.
Triangle C is not a reflection image of the red triangle.
Triangle D is the reflection image of the red triangle in the green line through the points (9, 1) and (1, 9).



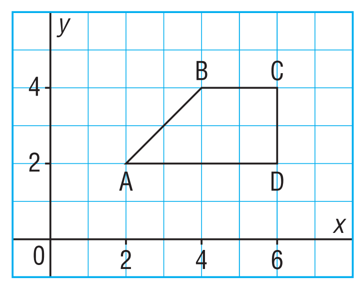
We can use a coordinate grid to draw shapes and their reflection images.

Example 3 Completing a Shape Given its Line of Symmetry

Quadrilateral ABCD is part of a larger shape.

- Draw the image of ABCD after each reflection below.
- Write the coordinates of the larger shape formed by ABCD and its image.
- Describe the larger shape and its symmetry.

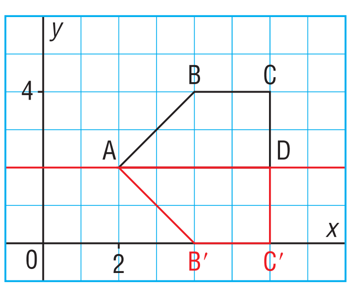
a) a reflection in the horizontal line through 2 on the y-axis
b) a reflection in the vertical line through 6 on the x-axis
c) a reflection in an oblique line through (0, 0) and (6, 6)



A Solution

The red line is the line of reflection. Each image point is the same distance from this line as the corresponding original point.

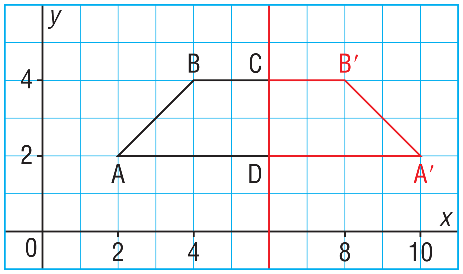
a)



Point	Image
A(2, 2)	A(2, 2)
B(4, 4)	B′(4, 0)
C(6, 4)	C′(6, 0)
D(6, 2)	D(6, 2)

The larger shape ABCC′B′ has coordinates: A(2, 2), B(4, 4), C(6, 4), C′(6, 0), B′(4, 0)
This shape is a pentagon with line symmetry. The line of symmetry is the red line.

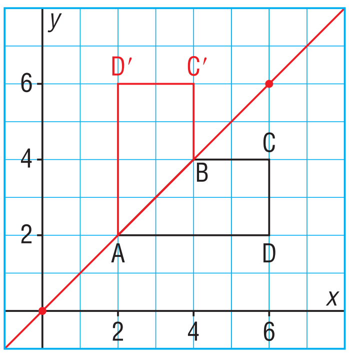
b)



Point	Image
A(2, 2)	A′(10, 2)
B(4, 4)	B′(8, 4)
C(6, 4)	C(6, 4)
D(6, 2)	D(6, 2)

The larger shape ABB′A′ has coordinates: A(2, 2), B(4, 4), B′(8, 4), A′(10, 2)
This shape is an isosceles trapezoid with line symmetry. The line of symmetry is the red line.

c)



Point	Image
A(2, 2)	A(2, 2)
B(4, 4)	B(4, 4)
C(6, 4)	C′(4, 6)
D(6, 2)	D′(2, 6)

The larger shape AD′C′BCD has coordinates: A(2, 2), D′(2, 6), C′(4, 6), B(4, 4), C(6, 4), D(6, 2)
This shape is a concave hexagon with line symmetry. The line of symmetry is the red line.

Discuss the ideas

1. How do you identify whether a shape has a line of symmetry?

2. How are a line of reflection and a line of symmetry related?

Practice

Check

3. You may have seen these hazardous substance warning symbols in the science lab. Which symbols have line symmetry? How many lines of symmetry?

a)

b)

c)

d)

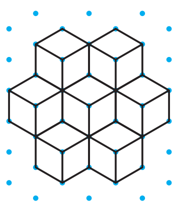
e)

f)

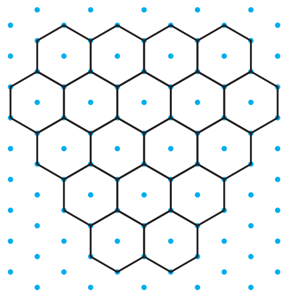
Apply

4. Identify the lines of symmetry in each tessellation.

a)

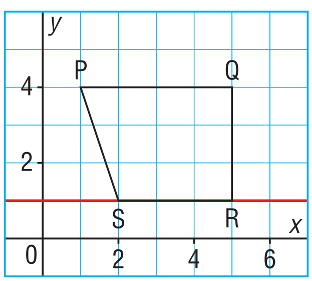


b)

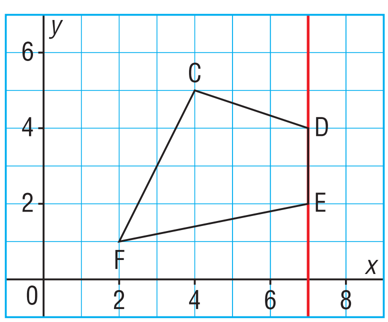


5. Copy each polygon on grid paper. It is one-half of a shape. Use the red line as a line of symmetry to complete the shape by drawing its other half. Label the shape with the coordinates of its vertices.

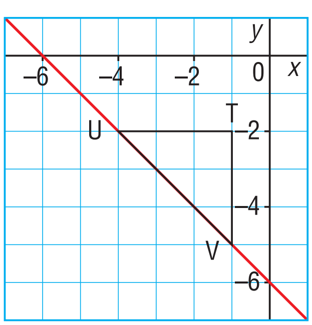
a)



b)



c)



6. State the number of lines of symmetry in each design.

a) a tessellation created by M.C. Escher

b) a Haida button blanket

7. **Assessment Focus**

a) Draw a triangle on a grid.

b) Choose one side of the triangle as a line of reflection.

i) Draw the reflection image.

ii) Label the vertices of the shape formed by the original triangle and its image.

iii) Write the coordinates of each vertex.

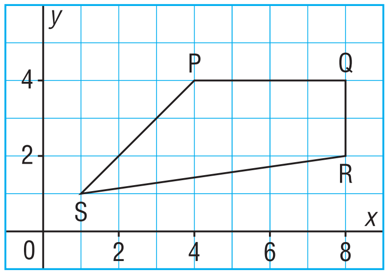
iv) How many lines of symmetry does the shape have?

c) Repeat part b for each of the other two sides of the triangle. Do you always get the same shape? Explain.

d) Repeat parts a to c for different types of triangles.

e) Which types of triangle always produce a shape that is a quadrilateral with line symmetry? Justify your answer.

8. Quadrilateral PQRS is part of a larger shape.



After each reflection below:

- Draw the image of PQRS.
- Write the coordinates of the larger shape formed by PQRS and its image.
- Describe the larger shape and its symmetry.

a) a reflection in the horizontal line through 4 on the y-axis

b) a reflection in the vertical line through 8 on the x-axis

c) a reflection in the oblique line through (1, 1) and (4, 4)

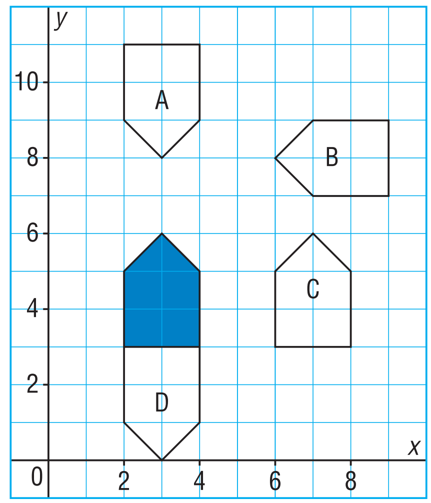
9. a) Graph these points on grid paper:
A(−3, 0), B(−1, 1), C(0, 3),
D(1, 1), E(3, 0).
Join the points to form polygon ABCDE.

b) Reflect the polygon in the x-axis. Draw and label its image.

c) Write the coordinates of the shape formed by the polygon and its image.

d) How many lines of symmetry does this shape have? How do you know?

10. Identify the pentagons that are related to the blue pentagon by a line of reflection. Describe the position of each line of symmetry.



Take It Further

11. a) On a grid, plot the points P(2, 2), Q(6, 2), and R(4, 4). Join the points to form ΔPQR.

b) Reflect ΔPQR in the line through the points (0, 4) and (4, 0). Draw the reflection image.

c) Reflect ΔPQR in the line through the points (0, −4) and (4, 0). Draw the reflection image.

d) Reflect ΔPQR in the x-axis. Draw the reflection image.

e) Look at the shape formed by the triangle and all its images. How many lines of symmetry does this shape have?

Reflect

When you see two shapes on a grid, how can you tell if they are related by a line of reflection?
Include examples of shapes that are related and are not related this way.

GAME

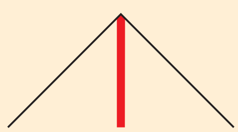
Make Your Own Kaleidoscope

The kaleidoscope was invented in 1816. It uses mirrors placed at different angles to produce patterns with symmetry.

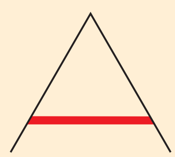
To make a simple kaleidoscope, use masking tape to join two mirrors so they stand at an angle.

Place your mirrors on the arms of each angle below.
Sketch and describe what you see.
Include any lines of symmetry in your sketch.

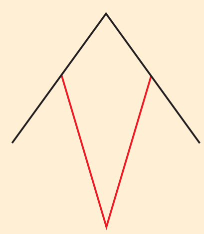
1.



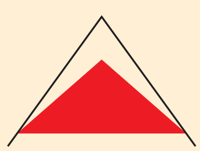
2.



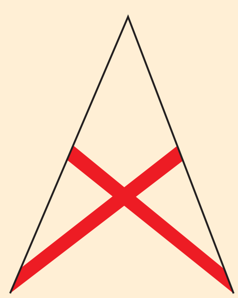
3.



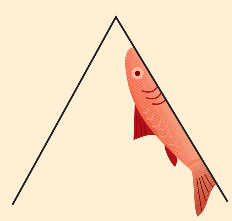
4.



5.



6.



You will need

- 2 small rectangular mirrors
- masking tape

7.6 Rotations and Rotational Symmetry

Look at these photographs.
How are the windmills the same?
How are they different?

FOCUS

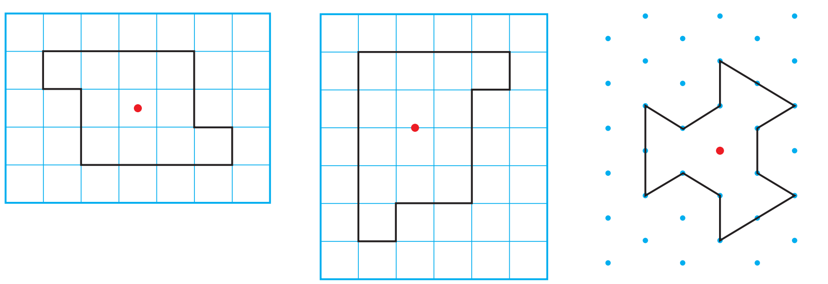
- Draw and classify shapes with rotational symmetry.

Investigate

3

You will need a protractor, a sharp pencil, tracing paper, and grid paper or isometric dot paper.

➤ Each of you chooses one of these shapes and copies it on grid paper or dot paper.



➤ Trace your shape and place the tracing to coincide with the shape.
Place a pencil point on the red dot.
Rotate the tracing, counting the number of times the tracing coincides with the original shape, until you make a complete turn.

➤ Repeat the rotation. This time, measure and record the angle you turned the tracing through each time.

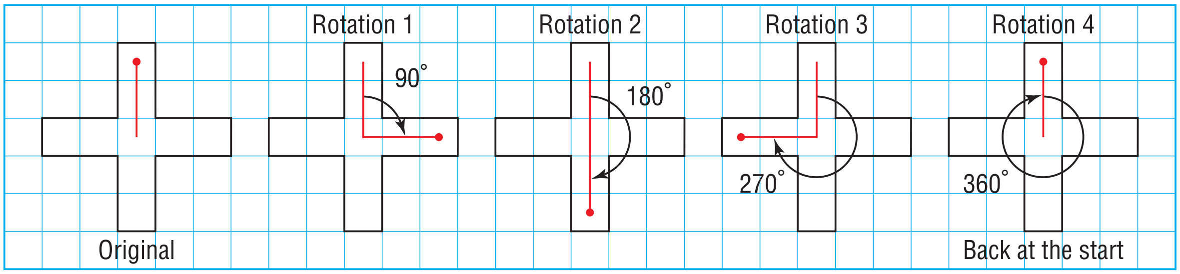
➤ Work together to draw a shape that coincides with itself 4 times as you rotate it.

Reflect & Share

Share your results with another group.
What is the relationship between the number of times the shape coincided with itself and the angle you turned it through each time?

Connect

A tracing of this shape is rotated about its centre. We draw a line segment to help identify the angle the shape turned through before it coincided with itself.



The shape coincided with itself 4 times in one complete turn; that is, during a rotation of 360°.

A shape has **rotational symmetry** when it coincides with itself after a rotation of less than 360° about its centre.
The number of times the shape coincides with itself, during a rotation of 360°, is the **order of rotation.** The shape above has *rotational symmetry of order* 4.

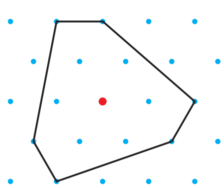
For each match, the shape rotated through 90°.
We say the **angle of rotation symmetry** is 90°. This is $\frac{360°}{4}$.

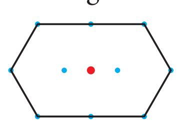
> In general, for rotational symmetry:
> the angle of rotation symmetry $= \frac{360°}{\text{the order of rotation}}$

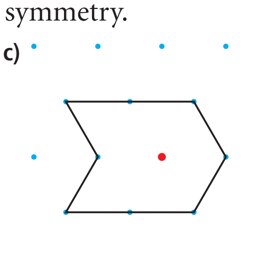
A shape that requires a rotation of 360° to return to its original position does not have rotational symmetry. A shape cannot have rotational symmetry of order 1.

Example 1 Identifying Shapes with Rotational Symmetry

Determine which hexagons below have rotational symmetry.
State the order of rotation and the angle of rotation symmetry.

a) 

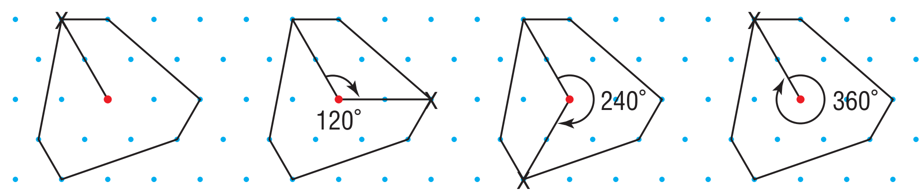
b) 

c) 

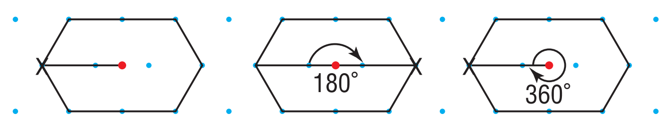
A Solution

For each hexagon:
- Join one vertex to the red dot.
- Trace the hexagon.
- Rotate the tracing about the red dot and record the order of rotation.
- Calculate the angle of rotation symmetry.

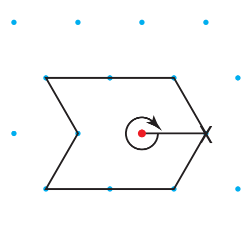
a) The order of rotation is 3.
The angle of rotation symmetry is: $\frac{360°}{3} = 120°$



b) The order of rotation is 2.
The angle of rotation symmetry is: $\frac{360°}{2} = 180°$



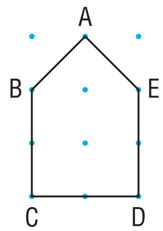
c) This hexagon is rotated one complete turn
before it coincides with itself.
It does not have rotational symmetry.



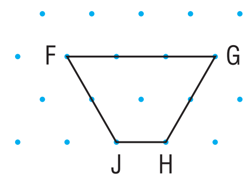
A rotation is another type of transformation.
We use a square grid to draw rotation images after a rotation of 90°, or any multiple of 90°, such as 180° and 270°.
We use isometric dot paper to draw rotation images after a rotation of 60°, or any multiple of 60°, such as 120° and 180°.

Example 2 Drawing Rotation Images

a) Rotate pentagon ABCDE 90° clockwise about vertex E. Draw the rotation image.



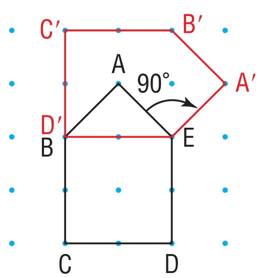
b) Rotate trapezoid FGHJ 120° counterclockwise about vertex F. Draw the rotation image.



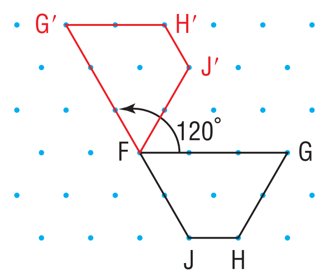
A Solution

Trace each shape and label the vertices on the tracing.

a) Rotate pentagon ABCDE 90° clockwise about E. Side ED moves from being vertical to being horizontal.



b) Rotate trapezoid FGHJ 120° counterclockwise about F. The angle between FG and FG′ is 120°.



Example 3 Identifying Symmetry after Rotations

a) Rotate rectangle ABCD:

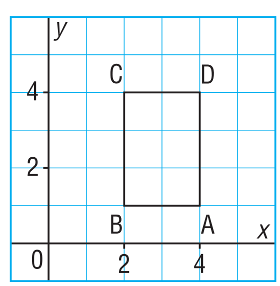
i) 90° clockwise about vertex A

ii) 180° clockwise about vertex A

iii) 270° clockwise about vertex A

Draw and label each rotation image.

b) Look at the shape formed by the rectangle and all its images. Identify any rotational symmetry in this shape.



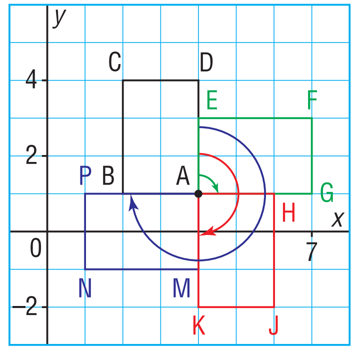
A Solution

a) Trace rectangle ABCD and label the vertices.

i) Rotate ABCD 90° clockwise about A.
Vertical side AD becomes horizontal side AG.
The rotation image is AEFG.

ii) Rotate ABCD 180° clockwise about A.
Vertical side AD becomes vertical side AK.
The rotation image is AHJK.

iii) Rotate ABCD 270° clockwise about A.
Vertical side AD becomes horizontal side AP.
The rotation image is AMNP.



b) The resulting shape BCDEFGHJKMNP has rotational symmetry of order 4 about point A.

Discuss the ideas

1. How do you determine whether a shape has rotational symmetry?

2. How can you determine:
a) the order of rotational symmetry?
b) the angle of rotation symmetry?

3. How is rotational symmetry related to rotation images?

Practice

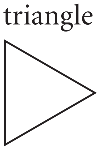
Check

4. What is the angle of rotation symmetry for a shape with each order of rotational symmetry?
a) 3 **b)** 5 **c)** 9 **d)** 12

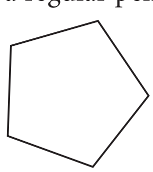
5. What is the order of rotational symmetry for each angle of rotation symmetry?
a) 60° **b)** 20° **c)** 45° **d)** 36°

6. What is the order of rotational symmetry and angle of rotation symmetry for each regular polygon?

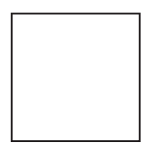
a) an equilateral triangle



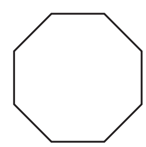
b) a regular pentagon



c) a square



d) a regular octagon

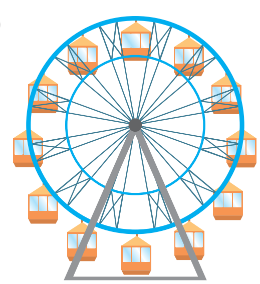


Apply

7. Does each picture have rotational symmetry? If it does, state the order and the angle of rotation symmetry.

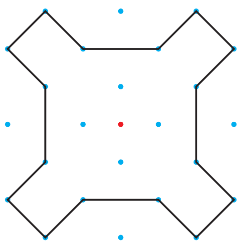
a)

b)

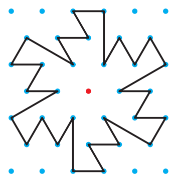


8. Does each shape have rotational symmetry about the red dot? If it does, state the order and the angle of rotation symmetry.

a)

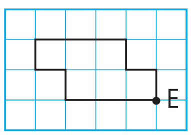


b)

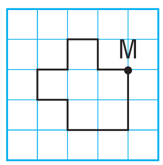


9. Copy each shape on grid paper. Draw the rotation image after each given rotation.

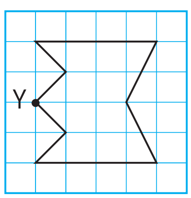
a) 90° clockwise about E



b) 180° about M

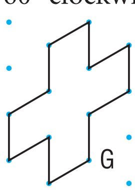


c) 270° counterclockwise about Y

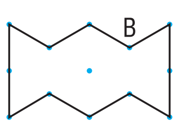


10. Copy each shape on isometric dot paper. Draw the rotation image after each given rotation.

a) 60° clockwise about G

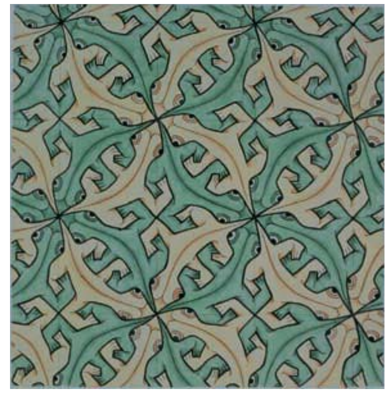


b) 120° counterclockwise about B

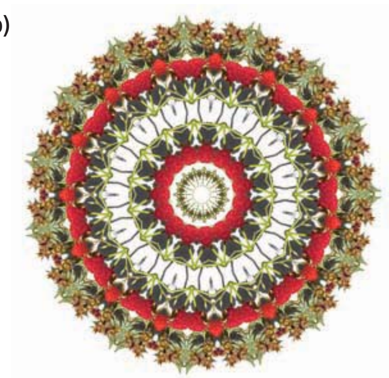


11. Identify and describe any rotational symmetry in each design.

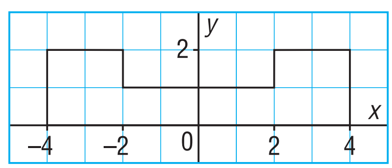
a)



b)



12. This octagon is part of a larger shape that is to be completed by a rotation of 180° about the origin.



a) On a coordinate grid, draw the octagon and its image.

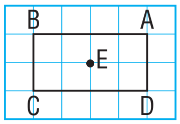
b) Outline the shape formed by the octagon and its image. Describe any rotational symmetry in this shape. Explain why you think the symmetry occurred.

13. Assessment Focus Rotate each shape.

a) rectangle ABCD

i) 180° about vertex A

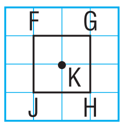
ii) 180° about centre E



b) square FGHJ counterclockwise through

i) 90° about vertex F

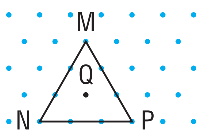
ii) 90° about centre K



c) equilateral triangle MNP clockwise through

i) 120° about vertex M

ii) 120° about centre Q

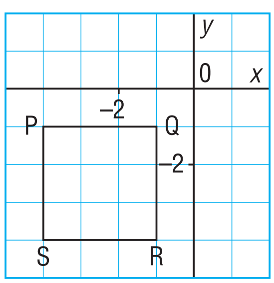


d) How are the images in each of parts a, b, and c the same? How are they different? Explain what you notice.

14. a) Rotate square PQRS clockwise about vertex P through:

i) 90° ii) 180° iii) 270°

Draw and label each rotation image.

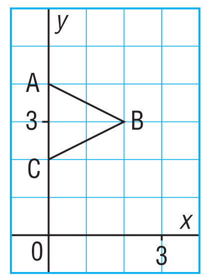


b) Outline the shape formed by the square and all its images. Identify any rotational symmetry. Explain what you notice.

15. Triangle ABC is part of a larger shape that is to be completed by three rotations.

a) Rotate ΔABC clockwise about vertex C through: i) 90° ii) 180° iii) 270°

Draw and label each rotation image.



b) List the coordinates of the vertices of the larger shape formed by the triangle and its images. Describe any rotational symmetry.

Take It Further

16. a) Draw a polygon on a coordinate grid. Choose an angle of rotation and a centre of rotation to complete a larger polygon with order of rotation: i) 2 ii) 4

List the coordinates of the centre of rotation, and the vertices of the larger polygon.

b) Draw a polygon on isometric dot paper. Choose an angle of rotation and a centre of rotation to complete a larger polygon with order of rotation: i) 3 ii) 6

Reflect

How do you decide if a given shape has rotational symmetry?
If it does, how do you determine the order of rotation and the angle of rotation symmetry?
Include an example in your explanation.

7.7 Identifying Types of Symmetry on the Cartesian Plane

FOCUS

- Identify and classify line and rotational symmetry.

What symmetry do you see in each picture?

Investigate

3

You will need grid paper and tracing paper.

➤ Plot these points on a coordinate grid: A(1, 3), B(3, 1), and C(5, 5)
Join the points to form ΔABC.

➤ Each of you chooses one of these transformations:
- a translation 2 units right and 2 units down
- a rotation of 180° about vertex C
- a reflection in a line through AB

Draw the image for the transformation you chose.
Record the coordinates of each vertex on the image.
On a separate piece of paper, record any symmetry in the triangle and its image.

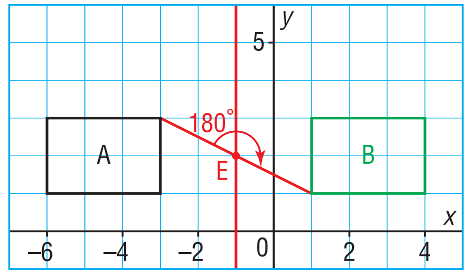
➤ Trade grids with a member of your group.
Identify any symmetry in the triangle and its image.

Reflect & Share

Compare the types of symmetry you found.
Did any grid show both rotational symmetry and line symmetry?
Explain why both types of symmetry occurred.
Which grid showed only one type of symmetry?

Connect

On this grid, rectangle A has been rotated 180° about E(-1, 2) to produce its image, rectangle B.
We can extend our meaning of line symmetry to relate the two rectangles.
The line through -1 on the x-axis is a line of symmetry for the two rectangles.
Each point on rectangle A has a corresponding point on rectangle B.
These points are equidistant from the line of symmetry.



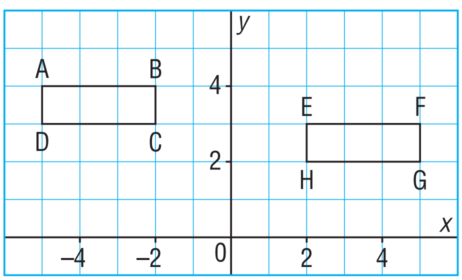
When a shape and its transformation image are drawn, the resulting diagram may show:

- no symmetry
- line symmetry
- rotational symmetry
- both line symmetry and rotational symmetry

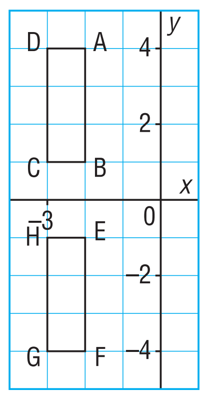
Example 1 Determining whether Shapes Are Related by Symmetry

For each pair of rectangles ABCD and EFGH, determine whether they are related by symmetry.

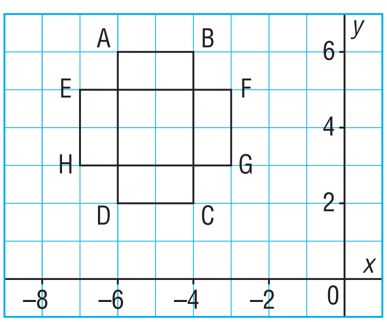
a)



b)

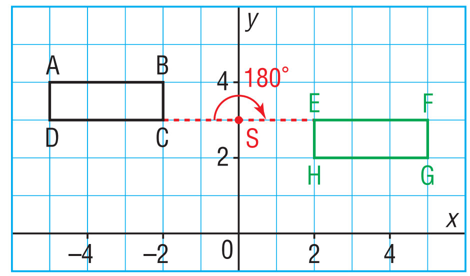


c)

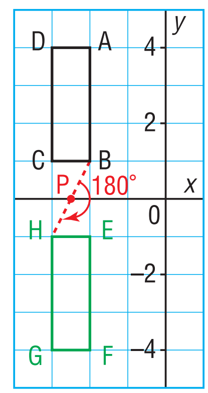


A Solution

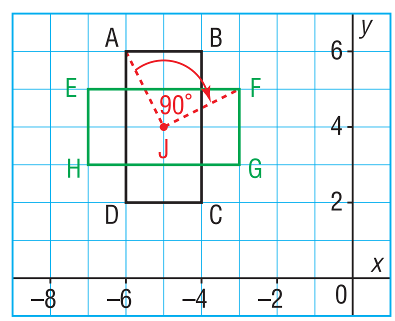
a) There is no line on which a mirror can be placed so that one rectangle is the reflection image of the other.
So, the rectangles are not related by line symmetry.
Trace the rectangles. Use guess and check to determine if a centre of rotation exists.
When ABCD is rotated 180° about the point S(0, 3), ABCD coincides with GHEF.
So, the rectangles are related by rotational symmetry of order 2 about S(0, 3).



b) Each point on ABCD has a corresponding point on EFGH.
These points are equidistant from the *x*-axis.
So, the two rectangles are related by line symmetry; the *x*-axis is the line of symmetry.
Trace the rectangles. Use guess and check to determine if a centre of rotation exists.
When a tracing of ABCD is rotated 180° about the point P(−2.5, 0), ABCD coincides with GHEF.
So, the two rectangles are related by rotational symmetry.



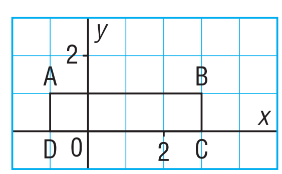
c) When ABCD is rotated 90° clockwise about point J(−5, 4), ABCD coincides with FGHE.
Then, the polygon formed by both rectangles together has rotational symmetry of order 4 about point J. So, the two rectangles are related by rotational symmetry.



Example 2 Identifying Symmetry in a Shape and Its Transformation Image

Draw the image of rectangle ABCD after each transformation.
Write the coordinates of each vertex and its image.
Identify and describe the type of symmetry that results.

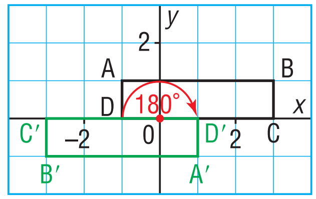
a) a rotation of 180° about the origin
b) a reflection in the *x*-axis
c) a translation 4 units right and 1 unit down



▶ ***Solution***

a) Use tracing paper to rotate ABCD 180° about the origin.

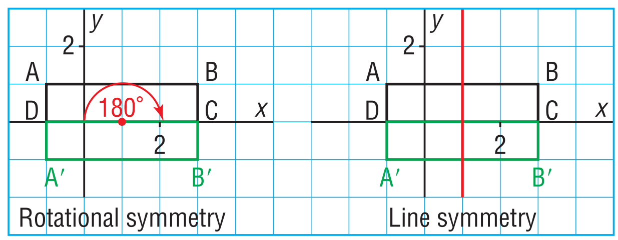
Point	Image
A(−1, 1)	A′(1, −1)
B(3, 1)	B′(−3, −1)
C(3, 0)	C′(−3, 0)
D(−1, 0)	D′(1, 0)



The octagon ABCD′A′B′C′D, formed by both rectangles together, has rotational symmetry of order 2 about the origin, and no line symmetry.

b) Reflect ABCD in the *x*-axis.

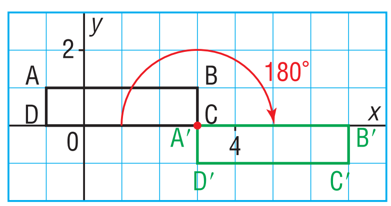
Point	Image
A(−1, 1)	A′(−1, −1)
B(3, 1)	B′(3, −1)
C(3, 0)	C(3, 0)
D(−1, 0)	D(−1, 0)



The rectangle ABB′A′, formed by both rectangles, has rotational symmetry of order 2 about the point (1, 0). It also has 2 lines of symmetry: the *x*-axis and the vertical line through 1 on the *x*-axis.

c) Translate ABCD 4 units right and 1 unit down.

Point	Image
A(−1, 1)	A′(3, 0)
B(3, 1)	B′(7, 0)
C(3, 0)	C′(7, −1)
D(−1, 0)	D′(3, −1)



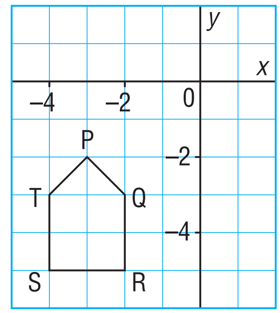
The two rectangles do not form a shape; but they have a common vertex at C (or A′). The two rectangles are related by rotational symmetry of order 2 about the point C(3, 0). There is no line of symmetry relating the rectangles.

In *Example 2*, we could write the translation 4 units right and 1 unit down in a shorter form as R4, D1. In this shorter form, a translation of 7 units left and 2 units up would be written as L7, U2.

Example 3 Identifying Symmetry in Shapes and their Translation Images

Draw the image of pentagon PQRST after each translation below.
Label the vertices of the pentagon and its image, and list their coordinates.
If each diagram has symmetry, describe it.
If each diagram does not have symmetry, explain how you know.

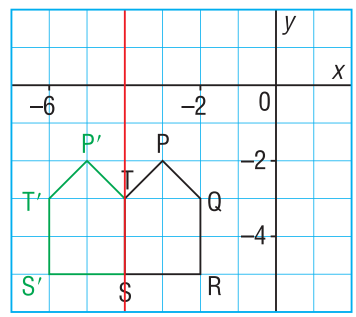
a) a translation L2 **b)** a translation L2, D3



A Solution

a) Translate each vertex of pentagon PQRST 2 units left.

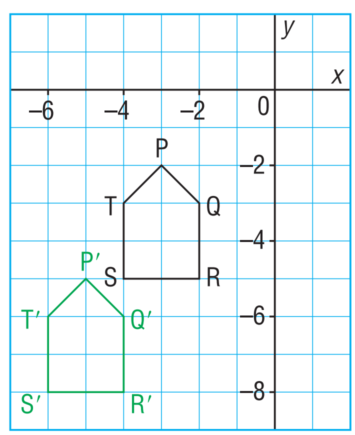
Point	Image
P(−3, −2)	P′(−5, −2)
Q(−2, −3)	T(−4, −3)
R(−2, −5)	S(−4, −5)
S(−4, −5)	S′(−6, −5)
T(−4, −3)	T′(−6, −3)



The diagram has line symmetry because the vertical line through ST is a line of reflection.
The diagram does not have rotational symmetry because there is no point about which it can be rotated so that it coincides with itself.

b) Translate each vertex of pentagon PQRST 2 units left and 3 units down.

Point	Image
P(−3, −2)	P′(−5, −5)
Q(−2, −3)	Q′(−4, −6)
R(−2, −5)	R′(−4, −8)
S(−4, −5)	S′(−6, −8)
T(−4, −3)	T′(−6, −6)



The diagram does not have line symmetry because there is no line on which a mirror can be placed so that one pentagon is the reflection image of the other.
The diagram does not have rotational symmetry because there is no point about which it can be rotated so that it coincides with itself.

Discuss the ideas

1. How can you tell if two shapes are related by line symmetry?

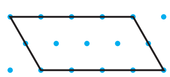
2. How can you tell if two shapes are related by rotational symmetry?

Practice

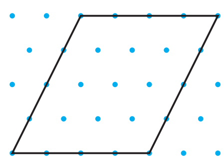
Check

3. Describe the rotational symmetry and line symmetry of each shape.

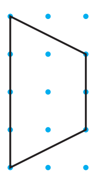
a) a parallelogram



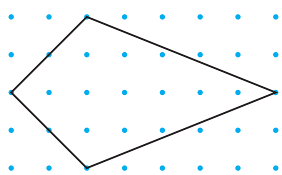
b) a rhombus



c) an isosceles trapezoid



d) a kite



4. Describe the rotational symmetry and line symmetry of each wheel cover. On a copy of the wheel covers, mark the centre of rotation and the line of reflection.

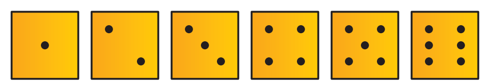
a)

b)

c)

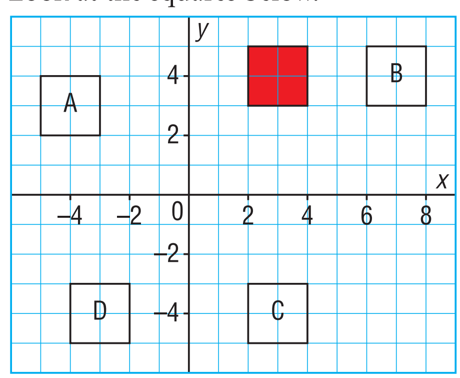
d)

5. Describe the symmetry of each face of a die. Copy each face. Mark the centre of rotation and the lines of symmetry.



Apply

6. Look at the squares below.



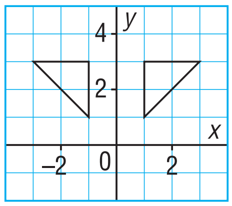
Which of squares A, B, C, and D are related to the red square:

a) by rotational symmetry about the origin?

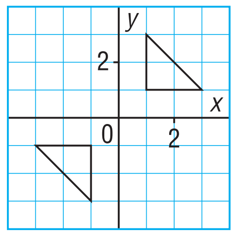
b) by line symmetry?

7. For each diagram, determine whether the two polygons are related by line symmetry, by rotational symmetry about the origin, or by both.

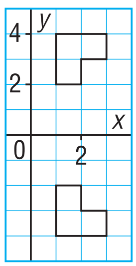
a)



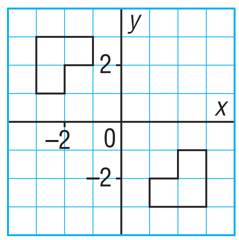
b)



c)

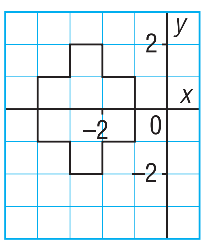


d)

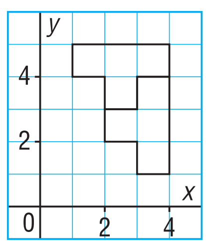


8. For each diagram, determine whether the two octagons are related by line symmetry, by rotational symmetry, by both types of symmetry, or by neither.

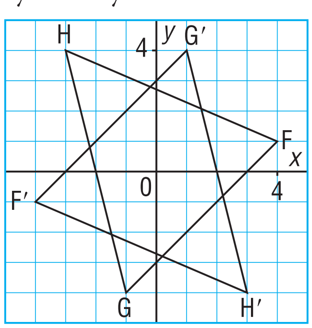
a)



b)



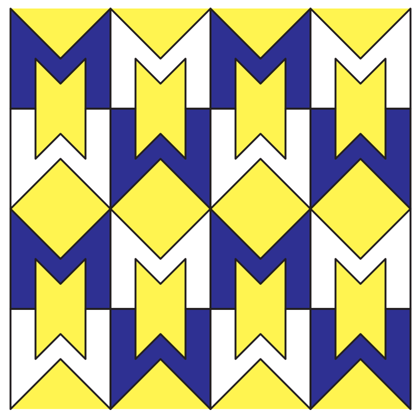
9. Triangle F′G′H′ is the image of ΔFGH after a rotation about the origin. Identify any symmetry.



10. Identify and describe the types of symmetry in each piece of artwork.

a)

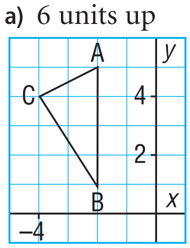
b)



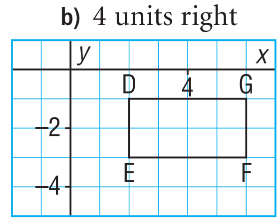
11. Copy each shape on grid paper.

- Draw the image after the translation given.
- Label each vertex with its coordinates.
- Does each diagram have line and rotational symmetry?
  If your answer is yes, describe the symmetry.
  If your answer is no, describe how you know.

a) 6 units up



b) 4 units right



12. **Assessment Focus**

a) On a grid, draw ΔCDE with vertices C(2, 3), D(−2, −1), and E(3, −2).

b) Draw the translation image $\Delta C'D'E'$ after the translation R1, U3.

c) Label all the vertices with their ordered pairs.

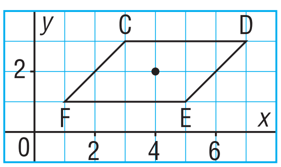
d) Explain why the translation does not result in line or rotational symmetry.

e) Find a translation that does result in one type of symmetry. Draw the image. How do you know the diagram has symmetry?

Show your work.

13. a) Draw the image of parallelogram CDEF after each transformation below.

b) The parallelogram and its image form a diagram. If each diagram has symmetry, describe it. If each diagram does not have symmetry, describe how you know.



i) a rotation of 90° clockwise about (4, 2)

ii) a reflection in the horizontal line through 1 on the y-axis

iii) a translation R4

14. The digits 0 to 9 on a digital clock are made up from horizontal and vertical segments.

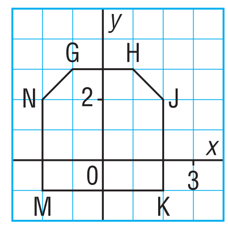
a) Sketch each digit on dot paper. Identify any symmetry it has.

b) For each digit with line symmetry, plot a part of the digit on grid paper and draw a line of symmetry so that the digit can be completed by a reflection.

c) For each digit with rotational symmetry, plot a part of the digit on grid paper. Locate the point about which the digit can be completed by a rotation.

d) Is there a pair of digits that are related by line or rotational symmetry? Justify your answer by plotting the digits on a Cartesian plane.

15. This hexagon is part of a larger shape that is completed by rotating the hexagon 180° about the origin.



a) Draw the rotation image.

b) List the coordinates of the vertices of the larger shape.

c) Describe the symmetry in the larger shape.

Take It Further

16. The 24-hour clock represents midnight as 00:00 and three-thirty A.M. as 03:30. The time 03:30 has line symmetry with a horizontal line of reflection. List as many times from midnight onward that have line symmetry, rotational symmetry, or both. Describe the symmetry for each time you find.

Reflect

When you see a shape and its transformation image on a grid, how do you identify line symmetry and rotational symmetry? Include examples in your explanation.

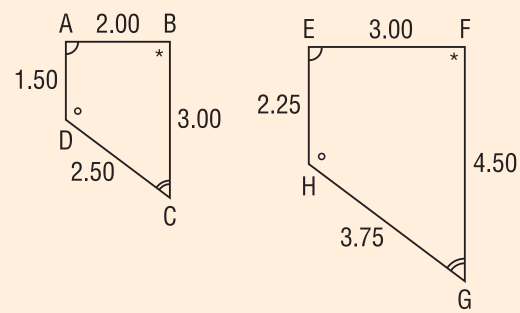
Study Guide

Scale Diagrams

For an enlargement or reduction, the scale factor is: $\dfrac{\text{Length on scale diagram}}{\text{Length on original diagram}}$

An enlargement has a scale factor > 1. A reduction has a scale factor < 1.

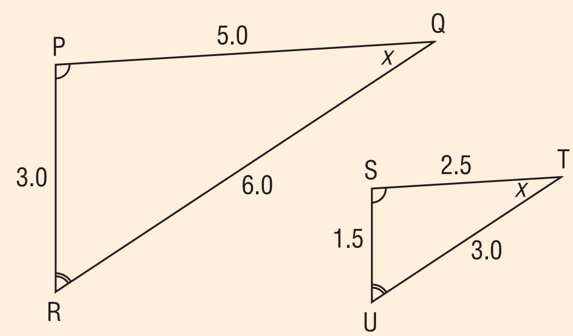
Similar Polygons



Similar polygons are related by an enlargement or a reduction. When two polygons are similar:

- their corresponding angles are equal:
  $\angle A = \angle E$; $\angle B = \angle F$; $\angle C = \angle G$; $\angle D = \angle H$
  and
- their corresponding sides are proportional:
  $\dfrac{AB}{EF} = \dfrac{BC}{FG} = \dfrac{CD}{GH} = \dfrac{DA}{HE}$
  Any of the ratios $\dfrac{AB}{EF}$, $\dfrac{BC}{FG}$, $\dfrac{CD}{GH}$, and $\dfrac{DA}{HE}$ is the scale factor.

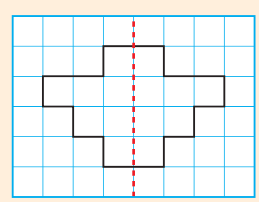
Similar Triangles



When we check whether two triangles are similar:

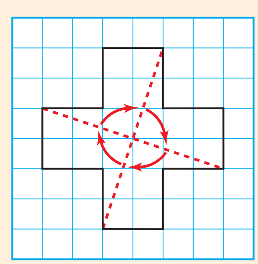
- their corresponding angles must be equal:
  $\angle P = \angle S$ and $\angle Q = \angle T$ and $\angle R = \angle U$
  or
- their corresponding sides must be proportional:
  $\dfrac{PQ}{ST} = \dfrac{QR}{TU} = \dfrac{PR}{SU}$
  Any of the ratios $\dfrac{PQ}{ST}$, $\dfrac{QR}{TU}$, and $\dfrac{PR}{SU}$ is the scale factor.

Line Symmetry



A shape has line symmetry when a line divides the shape into two congruent parts so that one part is the image of the other part after a reflection in the line of symmetry.

Rotational Symmetry



A shape has rotational symmetry when it coincides with itself after a rotation of less than 360° about its centre. The number of times the shape coincides with itself is the order of rotation.

The angle of rotation symmetry $= \dfrac{360°}{\text{the order of rotation}}$

Review

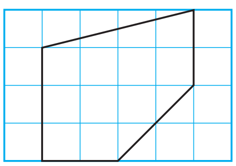
7.1

1. This photo of participants in the Arctic Winter Games is to be enlarged.

Measure the photo. What are the dimensions of the enlargement for each scale factor?

a) 3 **b)** 2.5 **c)** $\frac{3}{2}$ **d)** $\frac{21}{5}$

2. Draw this pentagon on 1-cm grid paper. Then draw an enlargement of the shape with a scale factor of 2.5.



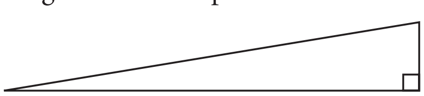
7.2

3. A full-size pool table has dimensions approximately 270 cm by 138 cm. A model of a pool table has dimensions 180 cm by 92 cm.

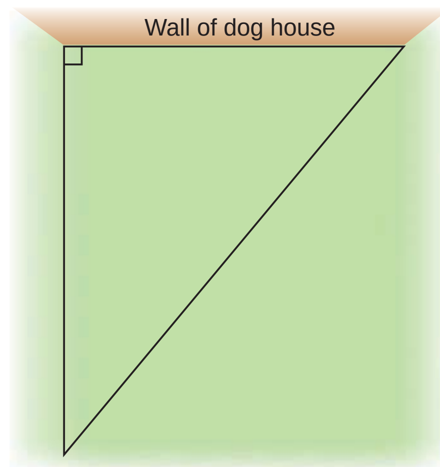
a) What is the scale factor for this reduction?

b) A standard-size pool cue is about 144 cm long. What is the length of a model of this pool cue with the scale factor from part a?

4. Here is a scale diagram of a ramp. The height of the ramp is 1.8 m. Measure the lengths on the scale diagram. What is the length of the ramp?

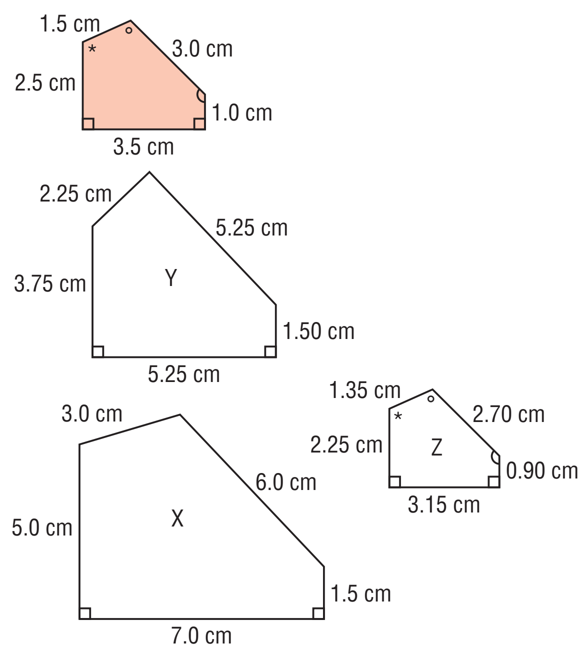


5. Gina plans to build a triangular dog run against one side of a dog house. Here is a scale diagram of the run. The wall of the dog house is 2 m long. Calculate the lengths of the other two sides of the dog run.

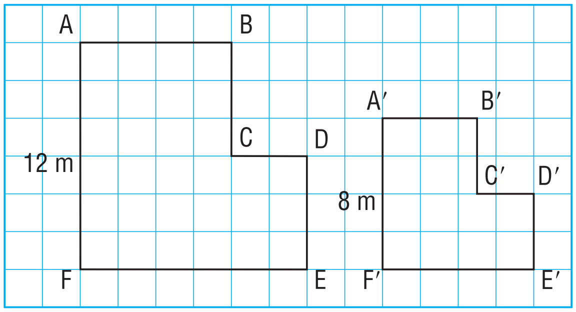


7.3

6. Which pentagon is similar to the red pentagon? Justify your answer.



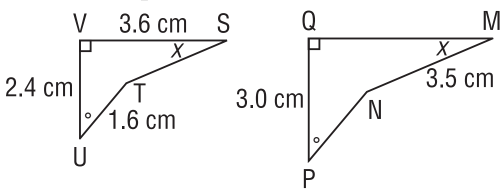
7. These two courtyards are similar.



Determine each length.

a) BC **b)** B′C′ **c)** A′B′

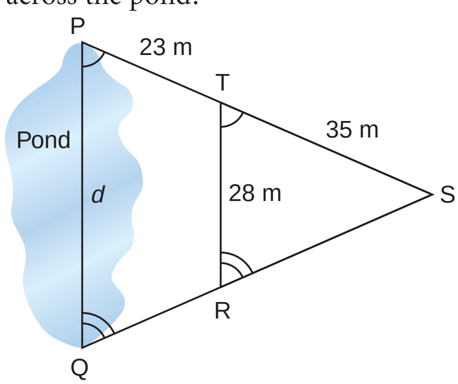
8. These two quadrilaterals are similar.



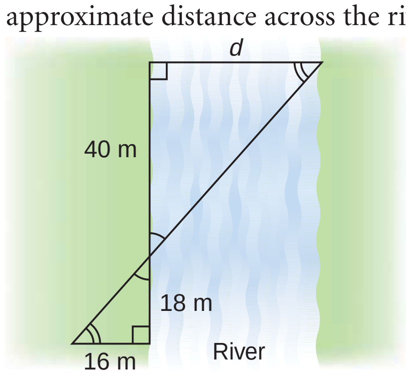
Calculate the length of: **a)** PN **b)** TS

7.4

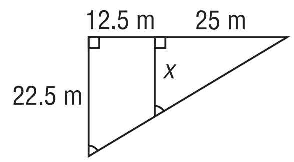
9. To determine the distance, d, across a pond, Ari uses this diagram. What is the distance across the pond?



10. This scale diagram shows a surveyor's measurements taken to determine the distance across a river. What is the approximate distance across the river?



11. How can you use similar triangles to calculate the distance x in this scale diagram?



7.5

12. Which of these traffic signs have line symmetry? How many lines of symmetry in each case?

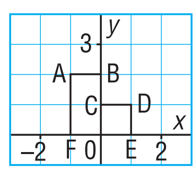
a)

b)

c)

d)

13. Hexagon ABCDEF is a part of a larger shape. Copy the hexagon on a grid.



a) Complete the shape by reflecting the hexagon:

i) in the y-axis

ii) in the x-axis

iii) in the line through $(-2, -1)$ and $(2, 3)$

b) Complete the shape with a translation R2.

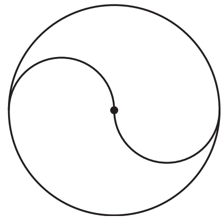
c) List the ordered pairs of the vertices of each completed shape.

d) State whether each completed shape has line symmetry.

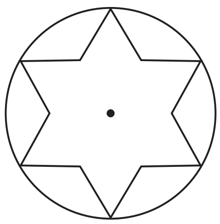
7.6 **14.** What is the order of rotational symmetry of each shape? How do you know?

a)

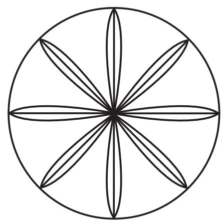
b)



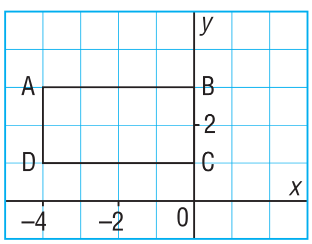
c)



d)



15. Rectangle ABCD is part of a larger shape that is to be completed by a transformation image.



a) Rotate rectangle ABCD as indicated, then draw and label each image.

i) 90° counterclockwise about the point $(-4, 2)$

ii) 180° about vertex B

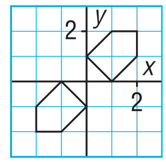
iii) 270° counterclockwise about the point $(-2, 2)$

b) Which diagrams in part a have rotational symmetry? How do you know?

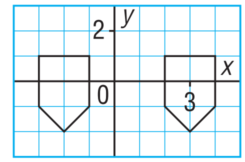
7.7 **16.** Look at the diagrams in question 15. Which diagrams have line symmetry? How do you know?

17. For each diagram, determine whether the two pentagons are related by any symmetry. Describe each type of symmetry.

a)



b)

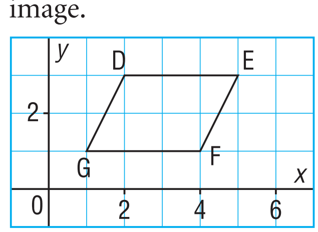


18. Identify and describe the types of symmetry in each piece of artwork.

a)

b)

19. a) Translate quadrilateral DEFG as indicated, then draw and label each image.

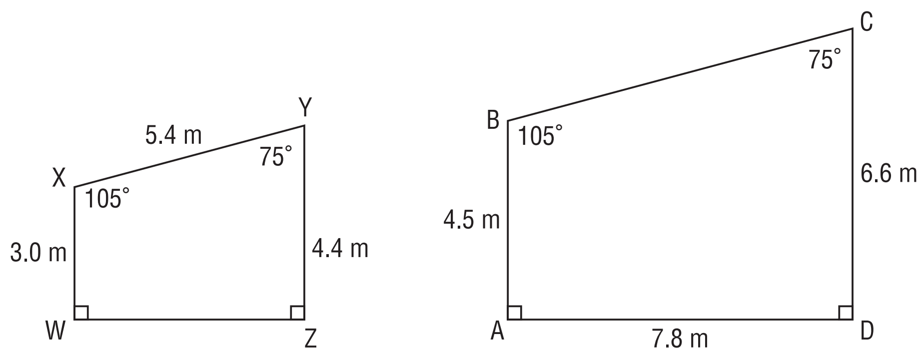


i) L4, D2 **ii)** R1, U2

b) Does each translation result in line symmetry or rotational symmetry? If your answer is yes, describe the symmetry. If your answer is no, explain why there is no symmetry.

Practice Test

1. These two quadrilaterals are similar.



a) Calculate the length of BC.
b) Calculate the length of WZ.
c) Draw an enlargement of quadrilateral WXYZ with scale factor 2.
d) Draw a reduction of quadrilateral ABCD with scale factor $\frac{1}{3}$.

2. Scott wants to calculate the height of a tree. His friend measures Scott's shadow as 3.15 m. At the same time, the shadow of the tree is 6.30 m. Scott knows that he is 1.7 m tall.
a) Sketch two triangles Scott could use to calculate the height of the tree.
b) How do you know the triangles are similar?
c) What is the height of the tree?

3. Use isometric dot paper or grid paper.
a) Draw these shapes: equilateral triangle, square, rectangle, parallelogram, trapezoid, kite, and regular hexagon
b) For each shape in part a:
  i) Draw its lines of symmetry.
  ii) State the order and angle of rotation symmetry.
c) Draw a shape that has line symmetry but not rotational symmetry.
d) Draw a shape that has rotational symmetry but not line symmetry.

4. Plot these points on a grid: A(2, 1), B(1, 2), C(1, 4), D(2, 5), E(3, 4), F(3, 2)
For each transformation below:
  i) Draw the transformation image.
  ii) Record the coordinates of its vertices.
  iii) Describe the symmetry of the diagram formed by the original shape and its image.
a) a rotation of 90° clockwise about the point G(2, 3)
b) a translation R2
c) a reflection in the line $y = 2$

Unit Problem — Designing a Flag

Part 1

At sea, flags are used to display messages or warnings. Here are some nautical flags.

➤ Describe the symmetries of each flag in as much detail as possible.

➤ Classify the flags according to the numbers of lines of symmetry.

Part 2

Design your own flag.
The flag may be for a country, an organization, or it may be a flag with a message. It must have line symmetry and rotational symmetry.

➤ Describe the symmetries in your flag.

The actual flag must be at least 3 m by 2 m.

➤ Draw a scale diagram of your flag, including the scale factor you used.

➤ Describe what your flag will be used for.

Your work should show:

- a description and classification of the symmetries of the nautical flags
- a scale diagram of your flag, in colour, including the scale factor
- a description of the symmetries in your flag
- a description of what your flag will be used for

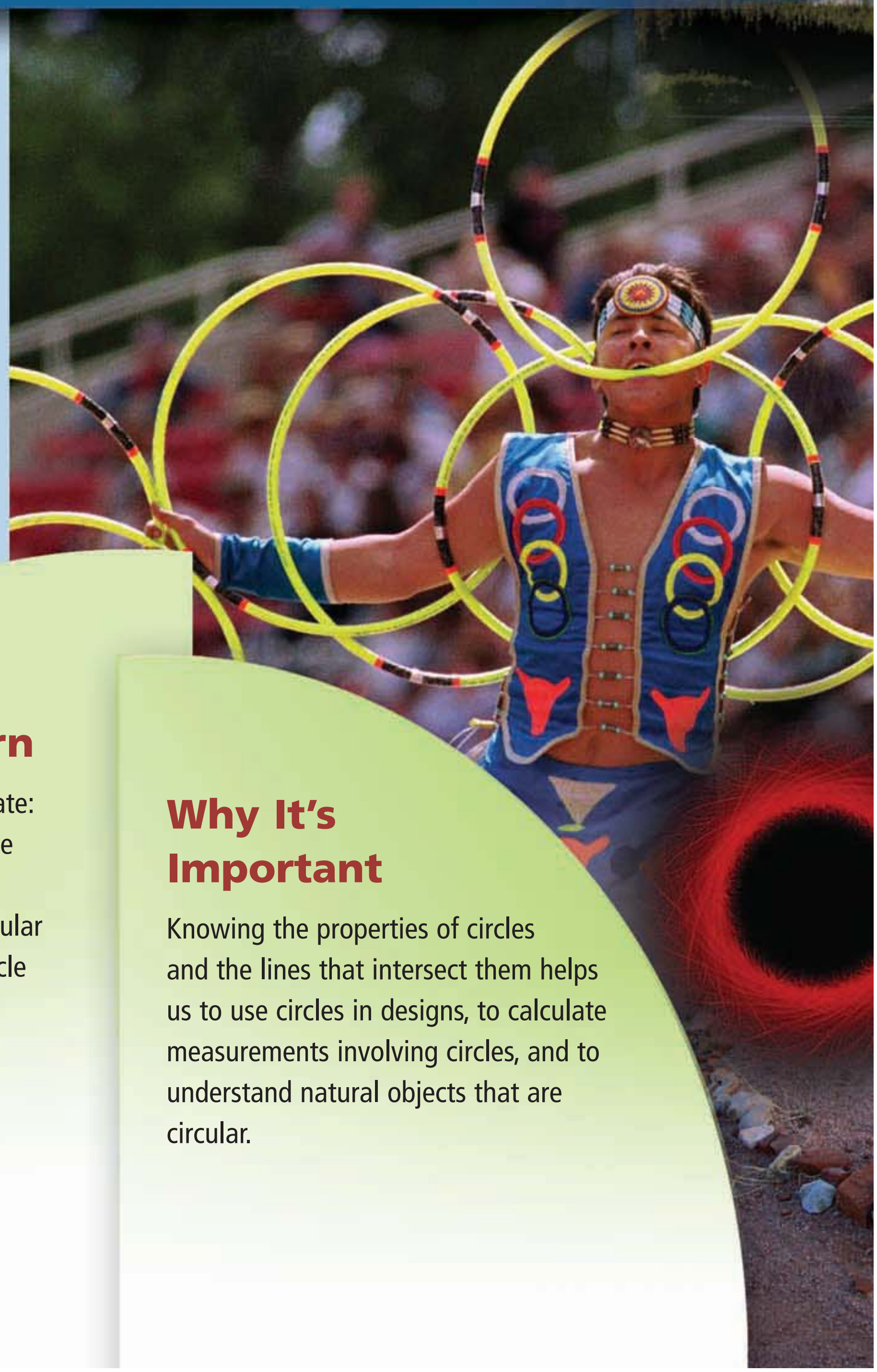
Reflect on Your Learning

How does knowledge of enlargements and reductions in scale diagrams help you understand similar polygons?
How are line symmetry and rotational symmetry related to transformations on a grid?

UNIT 8

Circle Geometry

We see circles in nature and in design.
What do you already know about circles?



What You'll Learn

Circle properties that relate:

- a tangent to a circle and the radius of the circle
- a chord in a circle, its perpendicular bisector, and the centre of the circle
- the measures of angles in circles

Why It's Important

Knowing the properties of circles and the lines that intersect them helps us to use circles in designs, to calculate measurements involving circles, and to understand natural objects that are circular.

Key Words

- tangent
- point of tangency
- chord
- arc
- minor arc
- major arc
- central angle
- inscribed angle
- subtended
- inscribed polygon
- supplementary angles

8.1 Properties of Tangents to a Circle

FOCUS

- Discover the relationship between a tangent and a radius, then solve related problems.

This wheel is rolling in a straight line on a flat surface.
The wheel touches the ground at only one point.
Visualize the red spoke extended to the ground.
What angle does the spoke appear to make with the ground?

Investigate

2

You will need a compass, protractor, and ruler.

➤ Construct a large circle and draw a radius.
Draw a line that touches the circle only at the endpoint of the radius.
Measure the angle between the radius and the line.

➤ Repeat the previous step for other radii and lines that touch the circle at an endpoint of a radius. Record your results.

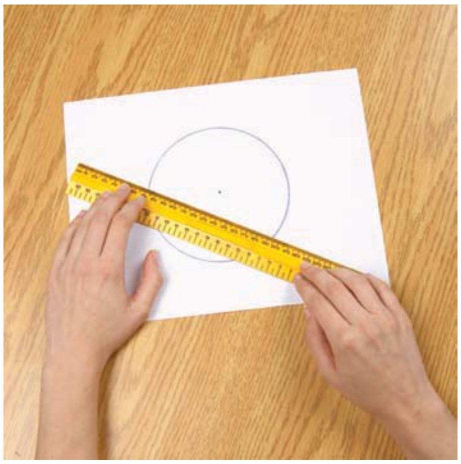
Repeat the procedure for other circles.
Write a statement about what you observe.

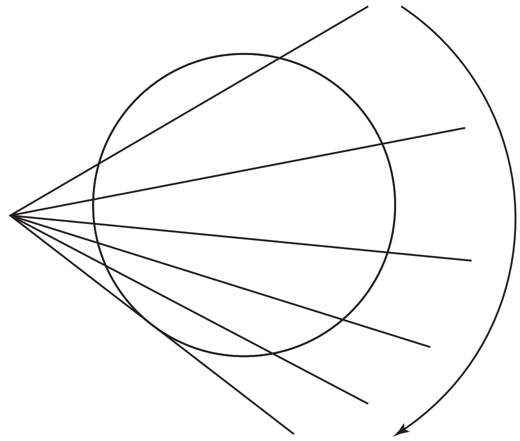
Reflect & Share

Compare your results with those of another pair of students.
What is the mean value of the angles you measured?
What do you think is the measure of the angle between the line you drew and the radius of each circle?

Connect

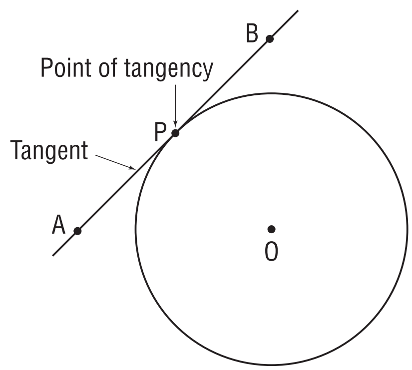
Imagine fixing one end of a ruler and rotating the ruler across a circle.





As one edge of the ruler sweeps across the circle, it intersects the circle at 2 points.
Just as the ruler leaves the circle, it intersects the circle at 1 point.
The edge of the ruler is then a tangent to the circle.

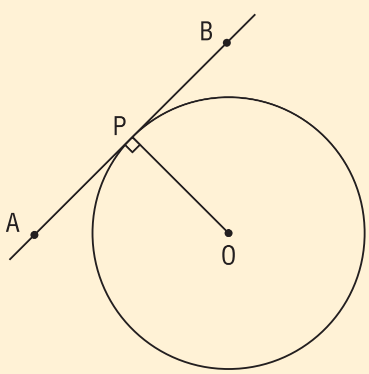
A line that intersects a circle at only one point is a **tangent** to the circle.
The point where the tangent intersects the circle is the **point of tangency**.



Line AB is a tangent to the circle with centre O.
Point P is the point of tangency.

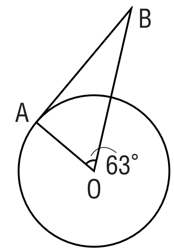
Tangent-Radius Property

A tangent to a circle is perpendicular to the radius at the point of tangency.
That is, $\angle APO = \angle BPO = 90°$



Example 1 Determining the Measure of an Angle in a Triangle

Point O is the centre of a circle
and AB is a tangent to the circle.
In ΔOAB, $\angle$AOB = 63°
Determine the measure of $\angle$OBA.

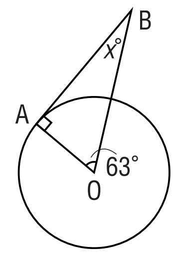


A Solution

Let $x°$ represent the measure of $\angle$OBA.
Since AB is a tangent to the circle, $\angle$OAB = 90°
The sum of the angles in ΔOAB is 180°.

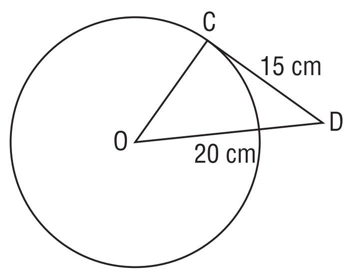
$$\text{So, } x° + 90° + 63° = 180°$$
$$x° = 180° - 90° - 63°$$
$$= 27°$$

So, $\angle$OBA = 27°



Example 2 Using the Pythagorean Theorem in a Circle

Point O is the centre of a circle
and CD is a tangent to the circle.
CD = 15 cm and OD = 20 cm
Determine the length of the radius OC.
Give the answer to the nearest tenth.

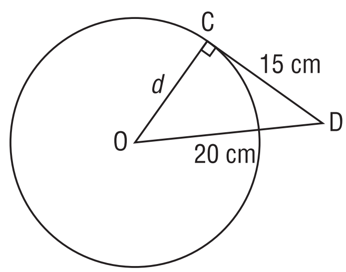


A Solution

Since CD is a tangent, $\angle$OCD = 90°
Use the Pythagorean Theorem in right ΔOCD to calculate OC.
Let d represent the length of OC.

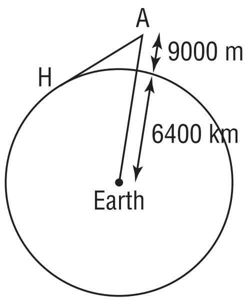
$$d^2 + \text{CD}^2 = \text{OD}^2$$
$$d^2 + 15^2 = 20^2$$
$$d^2 + 225 = 400$$
$$d^2 = 400 - 225$$
$$d^2 = 175$$
$$d = \sqrt{175}$$
$$d \doteq 13.23$$

The radius of the circle is about 13.2 cm long.



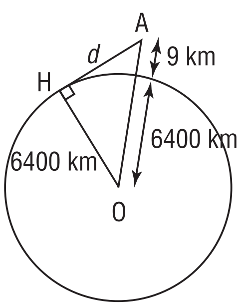
Example 3 Solving Problems Using the Tangent and Radius Property

An airplane, A, is cruising at an altitude of 9000 m.
A cross section of Earth is a circle with radius approximately 6400 km.
A passenger wonders how far she is from a point H on the horizon she sees outside the window.
Calculate this distance to the nearest kilometre.



A Solution

The line of sight AH from the passenger to the horizon is a tangent to the circle at H.



9000 m = 9 km

Since the tangent AH is perpendicular to the radius OH at the point of tangency H,
ΔAHO is a right triangle, with $\angle$OHA = 90°.
Use the Pythagorean Theorem to calculate AH.
Let d represent the length of AH.

$$d^2 + 6400^2 = (6400 + 9)^2$$
$$d^2 + 6400^2 = 6409^2$$
$$d^2 = 6409^2 - 6400^2$$
$$d = \sqrt{6409^2 - 6400^2}$$
$$d \doteq 339.53$$

The passenger is about 340 km from the horizon.

Discuss the ideas

1. A line may look as if it is a tangent to a circle but it may not be. How can you determine if the line is a tangent?

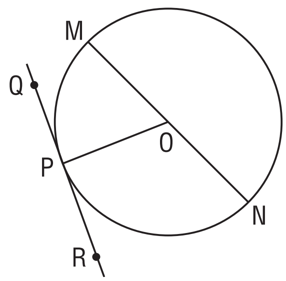
2. The Pythagorean Theorem was used in *Examples 2* and *3*. When is the Pythagorean Theorem useful for solving problems involving tangents?

Practice

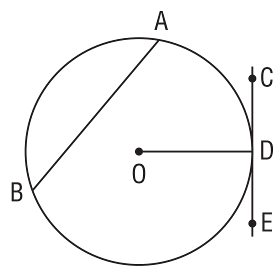
Check

3. In each diagram, point O is the centre of each circle. Which lines are tangents?

a)

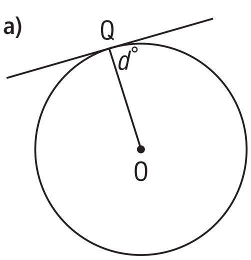


b)



4. Point Q is a point of tangency. Point O is the centre of each circle. What is each value of $d°$?

a)



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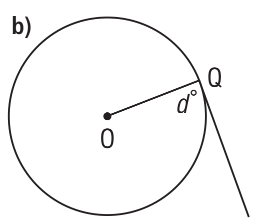
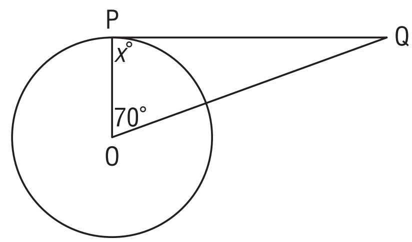


5. Point P is a point of tangency and O is the centre of each circle. Determine each value of $x°$.

a)



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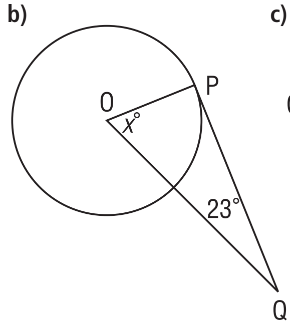
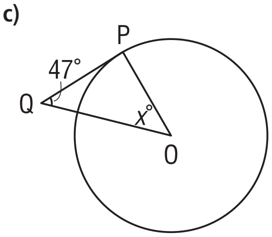
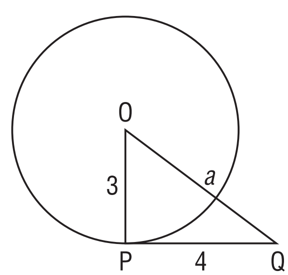


c)



6. Point P is a point of tangency and O is the centre of each circle. Determine each value of a.

a)



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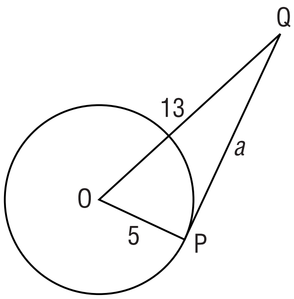

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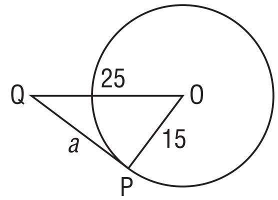
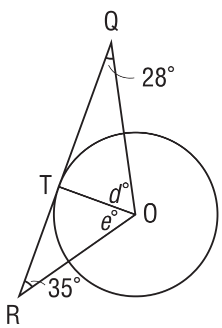


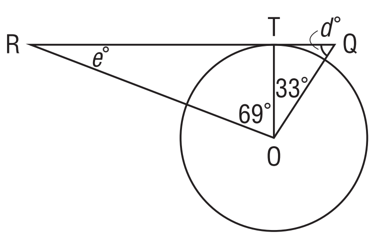
Apply

7. Point T is a point of tangency and O is the centre of each circle. Determine each value of $d°$ and $e°$.

a)

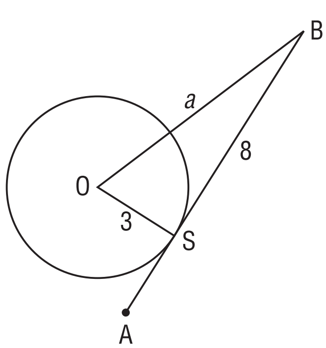


b)

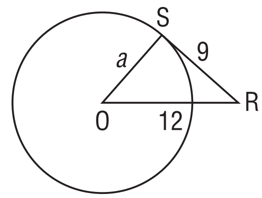


8. Point S is a point of tangency and O is the centre of each circle. Determine each value of a to the nearest tenth.

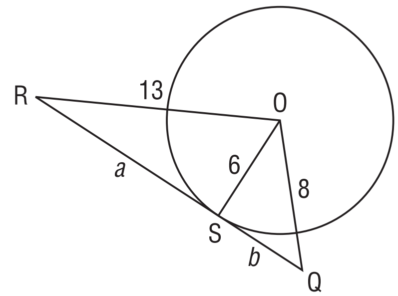
a)



b)

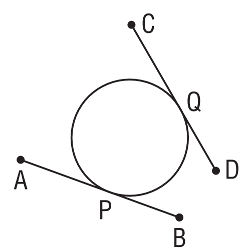


9. Point S is a point of tangency and O is the centre of the circle. Determine the values of a and b to the nearest tenth.

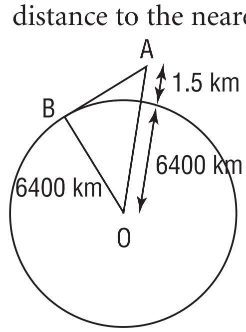


10. Look around the classroom or think of what you might see outside the classroom. Provide an example to illustrate that the tangent to a circle is perpendicular to the radius at the point of tangency.

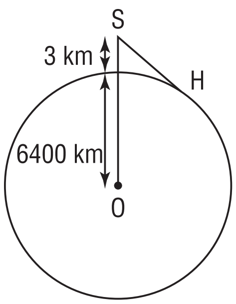
11. Both AB and CD are tangents to a circle at P and Q. Use what you know about tangents and radii to explain how to locate the centre of the circle. Justify your strategy.



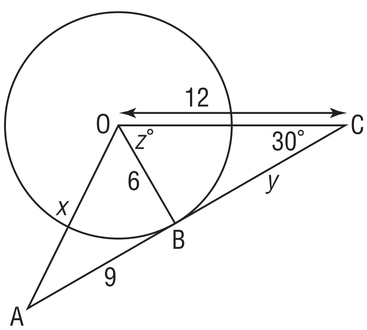
12. A small aircraft, A, is cruising at an altitude of 1.5 km. The radius of Earth is approximately 6400 km. How far is the plane from the horizon at B? Calculate this distance to the nearest kilometre.



13. A skydiver, S, jumps from a plane at an altitude of 3 km. The radius of Earth is approximately 6400 km. How far is the horizon, H, from the skydiver when she leaves the plane? Calculate this distance to the nearest kilometre.



14. Point O is the centre of the circle. Point B is a point of tangency. Determine the values of x, y, and $z°$. Give the answers to the nearest tenth where necessary. Justify the strategies you used.



15. Assessment Focus

a) From any point outside a circle, how many tangents do you think you can draw to the circle?
Explain your reasoning.

b) Construct a circle. Choose a point outside the circle. Check your answer to part a.
How do you know you have drawn as many tangents as you can?

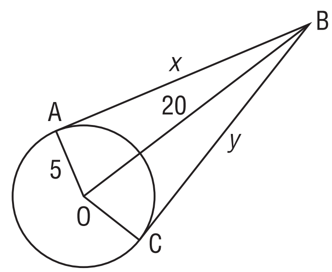
c) How do you know that the lines you have drawn are tangents?

Show your work.

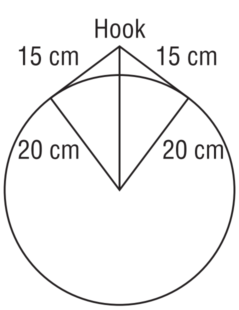
16. a) Construct a circle and draw two radii. Draw a tangent from the endpoint of each radius so the two tangents intersect at point N. Measure the distance from N to each point of tangency.
What do you notice?

b) Compare your answer to part a with that of your classmates. How do the lengths of two tangents drawn to a circle from the same point outside the circle appear to be related?

c) Points A and C are points of tangency and O is the centre of the circle. Calculate the values of x and y to the nearest tenth.
Do the answers confirm your conclusions in part b? Explain.



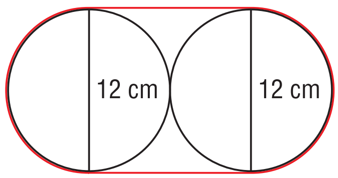
17. A circular mirror with radius 20 cm hangs by a wire from a hook. The wire is 30 cm long and is a tangent to the mirror in two places. How far above the top of the mirror is the hook? How do you know?



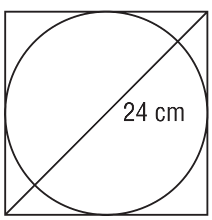
18. A communications satellite orbits Earth at an altitude of about 600 km. What distance from the satellite is the farthest point on Earth's surface that could receive its signal?
Justify the strategy you used.

Take It Further

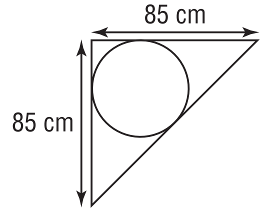
19. Two cylindrical rods are bound with a strap. Each rod has diameter 12 cm. How long is the strap? Give the answer to the nearest tenth of a centimetre. (The circumference C of a circle with diameter d is given by $C = \pi d$.)



20. What is the radius of the largest circle that can be cut from a square piece of paper whose diagonal is 24 cm long?



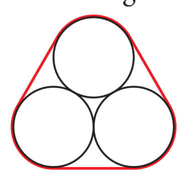
21. A cylindrical pipe touches a wall and the ceiling of a room. The pipe is supported by a brace. The ends of the brace are 85 cm from the wall and ceiling. Apply what you discovered in question 16. What is the diameter of the pipe? Give the answer to the nearest centimetre.



22. Each of 3 logs has diameter 1 m.

a) What is the minimum length of strap needed to wrap the logs?

b) Would this minimum length be the actual length of strap used? Explain.



Reflect

What do you know about a tangent and a radius in a circle? How can you use this property? Include examples in your explanation.

Math Link

Literacy

Sometimes a conversation goes off topic when the subject being discussed makes one person think of a related idea. For example, a discussion about Olympic athletes may prompt someone to think of and describe her exercise plan. When this happens, we say the discussion has "gone off on a tangent." How does this everyday occurrence relate to the meaning of the word "tangent" in math?

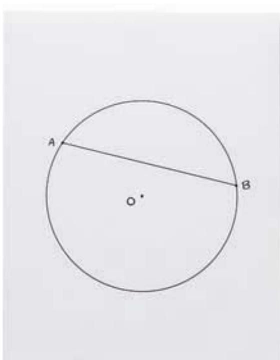
8.2 Properties of Chords in a Circle

FOCUS

- Relate a chord, its perpendicular bisector, and the centre of the circle, then solve problems.

These pictures show the sun setting.
Imagine the sun as it touches the horizon.
How is the centre of the sun related to the horizon?

Investigate



You will need scissors, a compass, protractor, and ruler.

➤ Construct then cut out a large circle. Label the centre of the circle O.

➤ Choose two points A and B on the circle. Join these points to form line segment AB. Make sure AB does *not* go through the centre of the circle.

➤ Fold the circle so that A coincides with B. Crease the fold, open the circle, and draw a line along the fold. Mark the point C where the fold line intersects AB.
What do you notice about the angles at C?
What do you notice about line segments AC and CB?

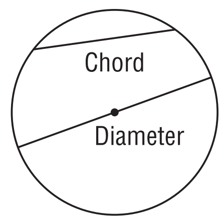
➤ Repeat the steps above for two other points D and E on the circle.

Reflect & Share

Compare your results with those of another pair of classmates.
What appears to be true about each line segment and its related fold line?
What name could you give each fold line?
Through which point do both fold lines appear to pass?

Connect

A line segment that joins two points on a circle is a **chord**.
A diameter of a circle is a chord through the centre of the circle.

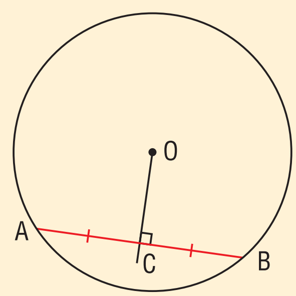


The chord, its perpendicular bisector, and the centre of the circle are related.

A perpendicular bisector intersects a line segment at 90° and divides the line segment into two equal parts.

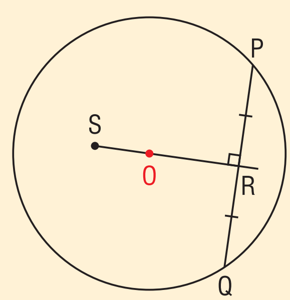
Perpendicular to Chord Property 1

The perpendicular from the centre of a circle to a chord bisects the chord; that is, the perpendicular divides the chord into two equal parts.
Point O is the centre of the circle.
When $\angle OCB = \angle OCA = 90°$,
then $AC = CB$



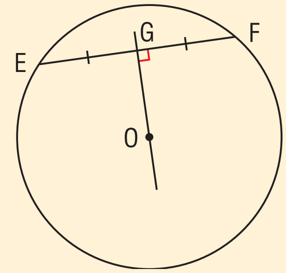
Perpendicular to Chord Property 2

The perpendicular bisector of a chord in a circle passes through the centre of the circle.
When $\angle SRP = \angle SRQ = 90°$ and $PR = RQ$,
then SR passes through O, the centre of the circle.



Perpendicular to Chord Property 3

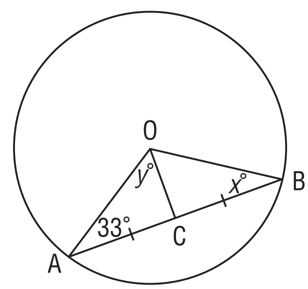
A line that joins the centre of a circle and the midpoint of a chord is perpendicular to the chord.
When O is the centre of a circle and EG = GF, then $\angle OGE = \angle OGF = 90°$



We can use these 3 properties to solve problems involving chords in a circle.

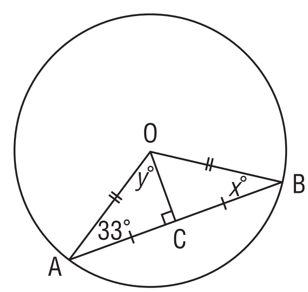
Example 1 Determining the Measure of Angles in a Triangle

Point O is the centre of a circle, and line segment OC bisects chord AB.
$\angle OAC = 33°$
Determine the values of $x°$ and $y°$.



A Solution

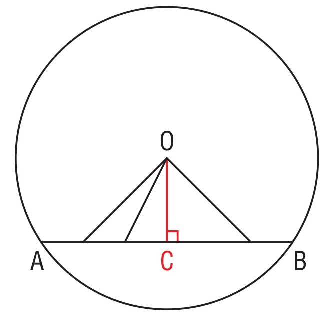
Since OC bisects the chord and passes through the centre of the circle, OC is perpendicular to AB.
So, $\angle ACO = 90°$
And, since the radii are equal, OA = OB, ΔOAB is isosceles.



Since ΔOAB is isosceles, then
$\angle OBA = \angle OAB$
So, $x° = 33°$
In ΔOAC, use the sum of the angles in a triangle.

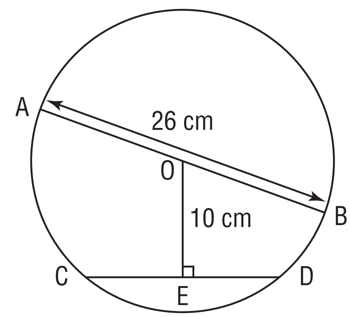
$$\begin{aligned} y° + 33° + 90° &= 180° \\ y° &= 180° - 90° - 33° \\ &= 57° \end{aligned}$$

Many line segments can be drawn from O, the centre of a circle, to a chord AB.
The *distance* from O to AB is defined as the shortest distance.
This distance is the length of the perpendicular from O to AB; that is, the length of OC.



Example 2 Using the Pythagorean Theorem in a Circle

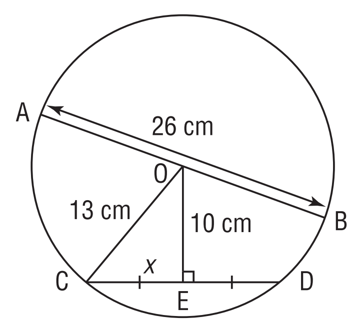
Point O is the centre of a circle.
AB is a diameter with length 26 cm.
CD is a chord that is 10 cm from the centre of the circle.
What is the length of chord CD?
Give the answer to the nearest tenth.



A Solution

The distance of a chord from the centre of a circle is the perpendicular distance from the centre to the chord.
Since OE is perpendicular to chord CD, then OE bisects CD, and CE = ED
To use the Pythagorean Theorem, join OC to form right ΔOCE.
OC is a radius, so OC is $\frac{1}{2}$ of AB, which is $\frac{1}{2}$ of 26 cm, or 13 cm.
Use the Pythagorean Theorem in ΔOCE to calculate CE.
Let the length of CE be represented by x.

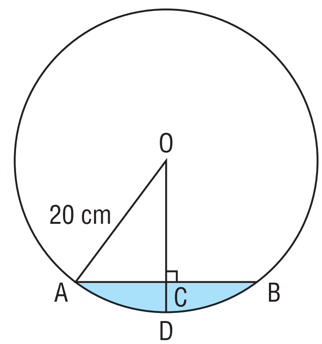
$$OC^2 = x^2 + OE^2$$
$$13^2 = x^2 + 10^2$$
$$169 = x^2 + 100$$
$$169 - 100 = x^2$$
$$69 = x^2$$
$$x = \sqrt{69}$$
$$x \doteq 8.307$$



So, CE $\doteq$ 8.307 cm
Chord CD = 2 $\times$ CE
$\doteq$ 2 $\times$ 8.307 cm
= 16.614 cm
Chord CD is about 16.6 cm long.

Example 3 Solving Problems Using the Property of a Chord and its Perpendicular

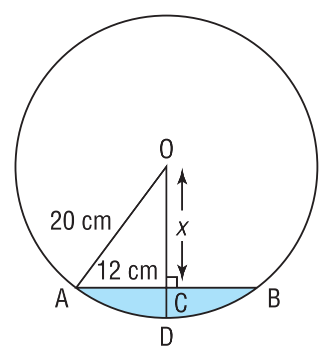
A horizontal pipe has a circular cross section, with centre O. Its radius is 20 cm.
Water fills less than one-half of the pipe.
The surface of the water AB is 24 cm wide.
Determine the maximum depth of the water, which is the depth CD.



A Solution

The depth CD = OD − OC
OD is the radius, 20 cm.
Since OC is perpendicular to AB, then AC = $\frac{1}{2}$ of AB,
which is $\frac{1}{2}$ of 24 cm, or 12 cm.
To determine OC, use the Pythagorean Theorem in ΔOAC.
Let x represent the length of OC.

$$AC^2 + x^2 = OA^2$$
$$12^2 + x^2 = 20^2$$
$$144 + x^2 = 400$$
$$x^2 = 400 - 144$$
$$x^2 = 256$$
$$x = \sqrt{256}$$
$$x = 16$$



$$\begin{aligned} CD &= OD - OC \\ &= 20\text{ cm} - 16\text{ cm} \\ &= 4\text{ cm} \end{aligned}$$

The maximum depth of the water is 4 cm.

Discuss the ideas

1. In a circle, how are these 3 items related?
- the centre of the circle
- a chord of a circle
- the perpendicular bisector of the chord

2. A diameter of a circle is a chord of the circle. How is the answer to question 1 affected if the chord is a diameter?

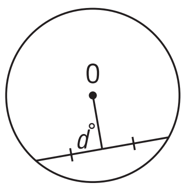
Practice

Check

Give the answers to the nearest tenth where necessary.

3. Point O is the centre of each circle. Determine the values of $d°$, e, and f.

a)



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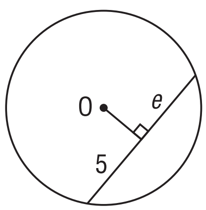

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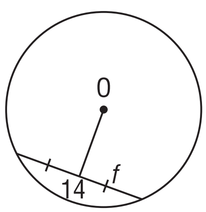
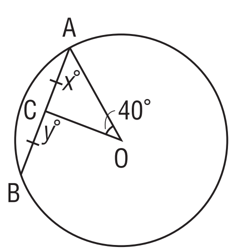
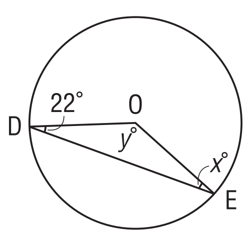


4. Point O is the centre of each circle. Determine each value of $x°$ and $y°$.

a)



b)



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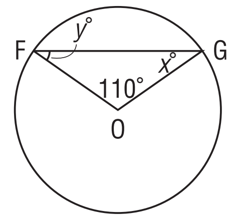
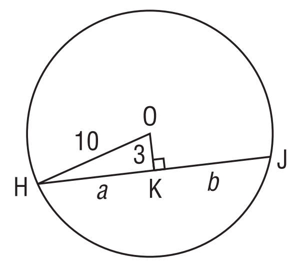


5. Point O is the centre of each circle. Determine each value of a and b.

a)



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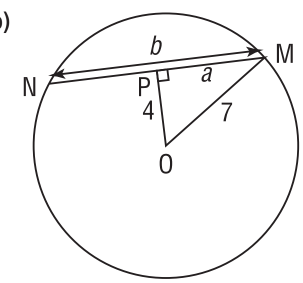
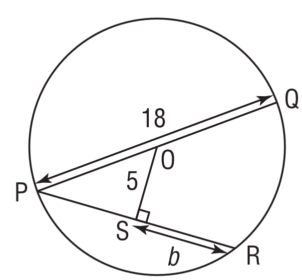


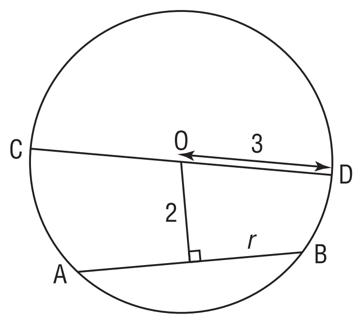
Apply

6. Point O is the centre of the circle. Determine the value of b. Which circle properties did you use?



7. Point O is the centre of each circle. Determine each value of r. Which extra line segments do you need to draw first? Justify your solutions.

a)



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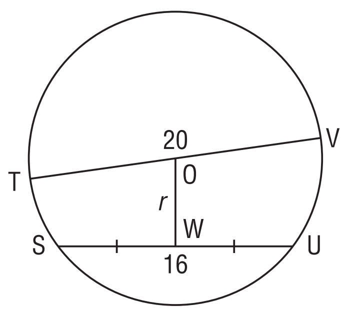
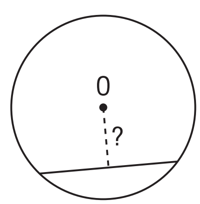


8. Construct a large circle, centre O.



a) Draw, then measure a chord in the circle. How far is the chord from O?

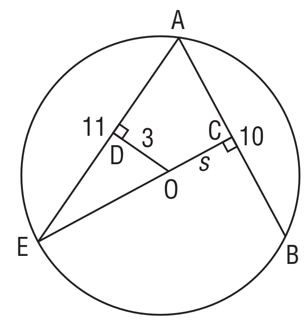
b) Draw other chords that are the same length as the chord you drew in part a. For each chord you draw, measure its distance from O. What do you notice?

c) Compare your results with those of other students. What appears to be true about congruent chords in a circle?

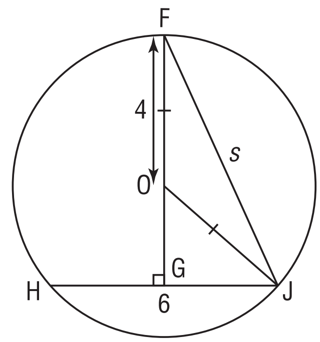
9. Trace a circular object to draw a circle without marking its centre. Draw two chords in the circle. Use what you have learned in this lesson to locate the centre of the circle. Justify your strategy.

10. Point O is the centre of each circle. Determine each value of s. Which circle properties did you use?

a)



b)



11. A circle has diameter 25 cm. How far from the centre of this circle is a chord 16 cm long? Justify your answer.

12. Assessment Focus

A circle has diameter 14 cm.

a) Which of the following measures could be lengths of chords in this circle? Justify your answers. How could you check your answers?

i) 5 cm **ii)** 9 cm

iii) 14 cm **iv)** 18 cm

b) For each possible length you identified in part a, determine how far the chord is from the centre of the circle.

Show your work. State which circle properties you used.

13. Draw and label a diagram to illustrate that the perpendicular to a chord from the centre of a circle bisects the chord.

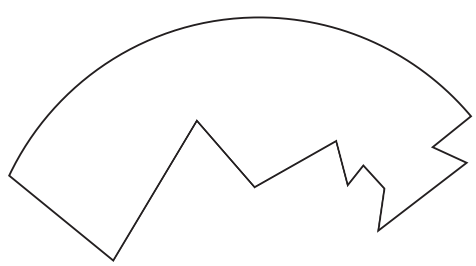
14. A chord is 6 cm long. It is 15 cm from the centre of a circle. What is the radius of the circle?

15. A circle has diameter 13 cm. In the circle, each of two chords is 8 cm long.

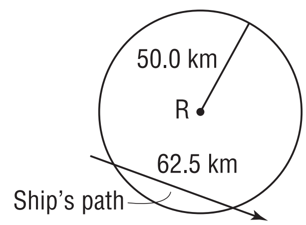
a) What is the shortest distance from each chord to the centre of the circle?

b) What do you notice about these congruent chords?

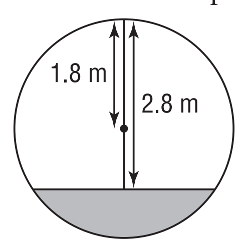
16. An archaeologist discovers a fragment of a circular plate on a dig at a prehistoric site. She wants to sketch the missing portion of the plate to determine how large it was. Trace the image of the plate fragment. Locate the centre of the plate. Use a compass to complete the sketch of the plate. Explain your work.



17. A radar station R tracks all ships in a circle with radius 50.0 km. A ship enters this radar zone and the station tracks it for 62.5 km until the ship passes out of range. What is the closest distance the ship comes to the radar station? Justify your answer.



18. A pedestrian underpass is constructed beneath a roadway using a cylindrical pipe with radius 1.8 m. The bottom of the pipe will be filled and paved. The headroom at the centre of the path is 2.8 m.
How wide is the path?



Take It Further

19. A spherical fish bowl has diameter 26 cm. The surface of the water in the bowl is a circle with diameter 20 cm.

a) What is the maximum depth of the water?

b) How many different answers are there for part a? Explain.

Reflect

What is the relationship among the centre of a circle, a chord, and the perpendicular bisector of the chord? Use an example to show how this relationship can help you calculate some measures in circles.

Verifying the Tangent and Chord Properties

FOCUS

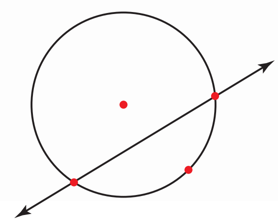
- Use dynamic geometry software to verify the tangent and chord properties in a circle.

Dynamic geometry software on a computer or a graphing calculator can be used to verify the circle properties in Lessons 8.1 and 8.2.
The diagrams show what you might see as you conduct the investigations that follow.

To verify the tangent property

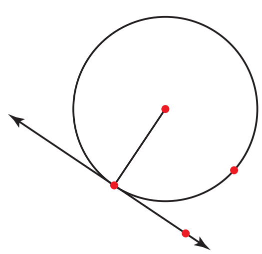
1. Construct a circle. If the software uses a point on the circle to define the circle, make sure you do not use this point for any further steps.

2. Construct a point on the circle.
If the software has a "Draw Tangent" tool, use it to draw a tangent at the point you constructed.



Otherwise, construct another point on the circle. Construct a line that intersects the circle at the two points.
Drag one of the two points on the line until it coincides with the other point. The line is now a tangent to the circle.

3. Construct a line segment to join the centre of the circle and the point of tangency. What does this line segment represent?



4. To measure the angle between the tangent and radius, you need a second point on the tangent. Construct a second point if the software does not do this automatically.

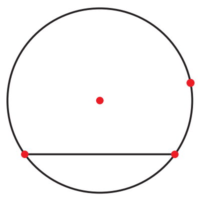
5. Use the software's measurement tools to measure the angle between the radius and tangent. Does the angle measure match what you have learned about a tangent and radius? If not, suggest a reason why.

6. Draw other tangents to the circle and the radii that pass through the points of tangency. Measure the angle between each tangent and radius. What do you notice?

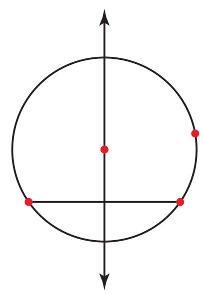
7. Drag either the circle or its centre to investigate the property for circles of different sizes. What is always true about the angle between a tangent to a circle and a radius at the point of tangency?

To verify the chord property

1. Construct a circle.
2. Construct a line segment to join two points on the circle.



3. Construct a line perpendicular to the segment through the centre of the circle.



4. Use measurement tools to measure the distance between each endpoint of the chord and the point of intersection of the chord and the perpendicular. What do you notice?
5. Drag the endpoints of the chord to different positions on the circle to check the results for other chords. What do you notice?
6. Drag the circle or its centre to investigate the property for circles of different sizes. What is always true about a perpendicular from the centre of a circle to a chord in the circle?

Check

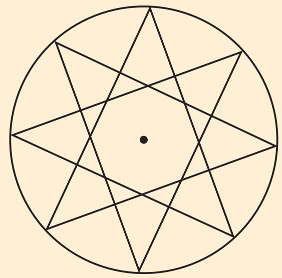
1. Suppose you have constructed a circle and one of its radii. Suggest a way, different from those mentioned above, to construct a tangent at the endpoint of the radius. Construct several circles, radii, and tangents to demonstrate your ideas.
2. Use the geometry software to verify these properties of chords.
   a) Construct a chord. Construct its midpoint. Construct a perpendicular through the midpoint. Through which point does the perpendicular bisector of the chord pass?
   b) Construct a chord. Construct its midpoint.
   Construct a line segment from the midpoint to the centre of the circle. What is the measure of the angle between the chord and this line segment?

GAME

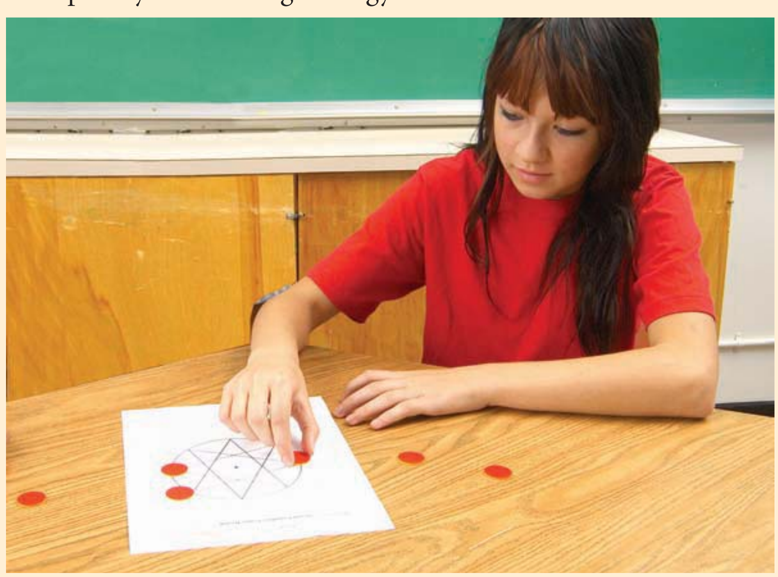
Seven Counters

How to Play

Your teacher will give you a larger copy of this game board:



1. Place a counter on a vertex of the star in the game board. Slide the counter along any segment of the star to place it at another vertex. The counter is now fixed.
2. Continue placing counters by sliding them from one vertex to another until no more counters can be placed. A counter cannot be placed on the game board if no vertex has a line segment along which the counter can move.
3. If you play with a partner, take turns to place counters and work together to decide on a winning strategy.
4. If you play against a partner, work independently to see who can place more counters.
5. It is possible to place all seven counters on the board, leaving one vertex of the star open.
   Keep trying until you have succeeded!
6. Explain your winning strategy.



You will need

- a copy of the game board
- 7 counters

Number of Players

- 1 or 2

Goal of the Game

- To place all seven counters on the game board

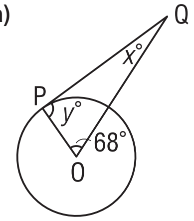
Mid-Unit Review

Give the answers to the nearest tenth where necessary.

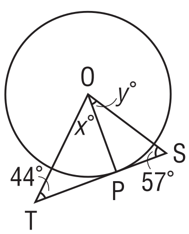
8.1

1. Point O is the centre of each circle and P is a point of tangency. Determine each value of $x°$ and $y°$.
Which circle properties did you use?

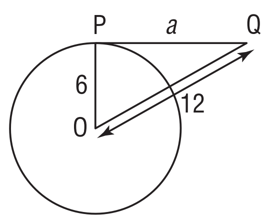
a)



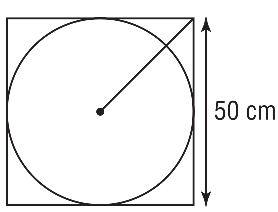
b)



2. Point O is the centre of a circle and point P is a point of tangency. Determine the value of a. Explain your strategy.

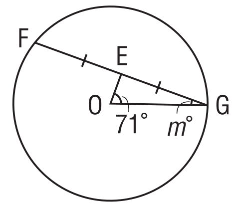


3. A metal disc is to be cut from a square sheet with side length 50 cm.
How far from a corner of the sheet is the centre of the disc? Justify your strategy.



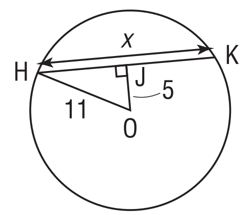
8.2

4. Point O is the centre of the circle.
Determine the value of $m°$.



5. Point O is the centre of each circle.
Determine each value of x.

a)



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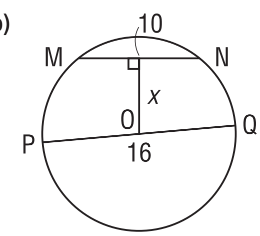
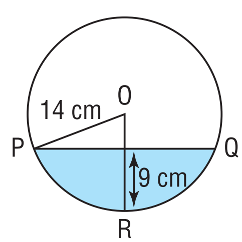


6. A circle has diameter 32 cm. A chord AB is 6 cm from O, the centre of the circle.

a) Sketch a diagram.

b) What is the length of the chord? Which circle properties did you use to find out?

7. Water is flowing through a pipe with radius 14 cm. The maximum depth of the water is 9 cm. What is the width, PQ, of the surface of the water?



8.3 Properties of Angles in a Circle

FOCUS

- Discover the properties of inscribed angles and central angles, then solve related problems.

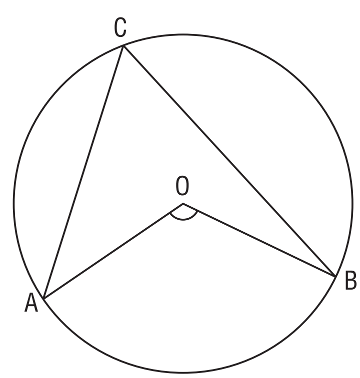
A soccer player attempts to get a goal. In a warm-up, players line up parallel to the goal line to shoot on the net. Does each player have the same shooting angle? Is there an arrangement that allows the players to be spread out but still have the same shooting angle?

Investigate

2

You will need a compass, ruler, and protractor.

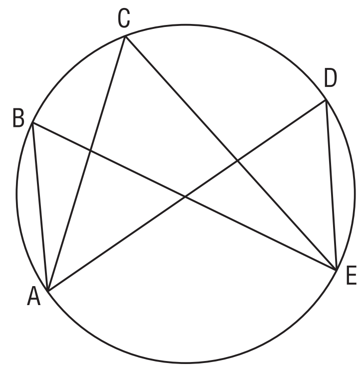
➤ Construct a large circle, centre O.
Choose two points A and B on the circle.
Choose a third point C on the circle.
Join AC and BC.
Measure ∠ACB. Join AO and OB.
Measure the smaller ∠AOB.
Record your measurements.



➤ Repeat the previous step for other points A, B, and C on the circle and for other circles.

➤ Construct another large circle.
Mark 5 points A, B, C, D, and E, in order, on the circle.
Join AB, AC, AD, and EB, EC, ED.
Measure ∠ABE, ∠ACE, and ∠ADE.
Record your measurements.

➤ Repeat the previous step for other circles.

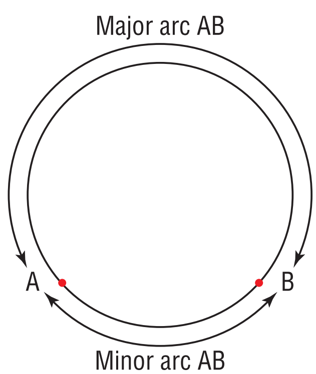


Reflect & Share

Compare your results with those of another pair of students.
What relationship did you discover between the angle at the centre of a circle and the angle on the circle?
What relationship did you discover among the angles on a circle?

Connect

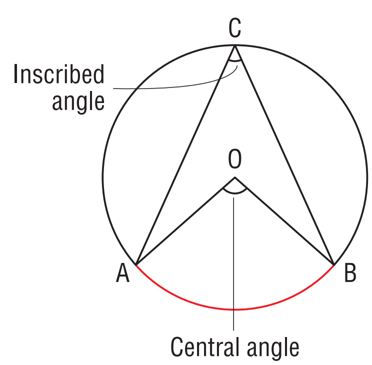
➤ A section of the circumference of a circle is an **arc**.
The shorter arc AB is the **minor arc**.
The longer arc AB is the **major arc**.



➤ The angle formed by joining the endpoints of an arc to the centre of the circle is a **central angle**;
∠AOB is a central angle.

The angle formed by joining the endpoints of an arc to a point on the circle is an **inscribed angle**;
∠ACB is an inscribed angle.

The inscribed and central angles in this circle are **subtended** by the minor arc AB.

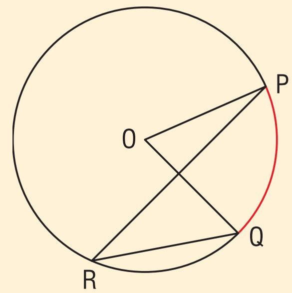


Central Angle and Inscribed Angle Property

In a circle, the measure of a central angle subtended by an arc is twice the measure of an inscribed angle subtended by the same arc.

$\angle POQ = 2\,\angle PRQ$, or

$\angle PRQ = \frac{1}{2}\,\angle POQ$

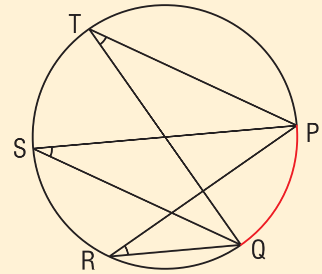


The above property is true for any inscribed angle.

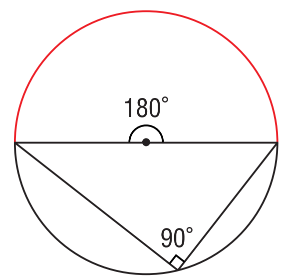
Inscribed Angles Property

In a circle, all inscribed angles subtended by the same arc are congruent.

$\angle PTQ = \angle PSQ = \angle PRQ$



➤ The two arcs formed by the endpoints of a diameter are semicircles.
The central angle of each arc is a straight angle, which is 180°.
The inscribed angle subtended by a semicircle is one-half of 180°, or 90°.

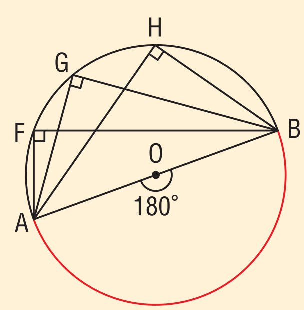


Angles in a Semicircle Property

All inscribed angles subtended by a semicircle are right angles.

Since $\angle AOB = 180°$,

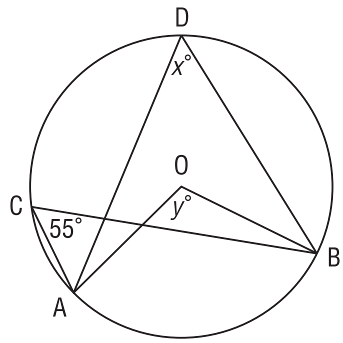
then $\angle AFB = \angle AGB = \angle AHB = 90°$



We say: The angle *inscribed* in a semicircle is a right angle.
We also know that if an inscribed angle is 90°, then it is subtended by a semicircle.

Example 1 Using Inscribed and Central Angles

Point O is the centre of a circle.
Determine the values of $x°$ and $y°$.



A Solution

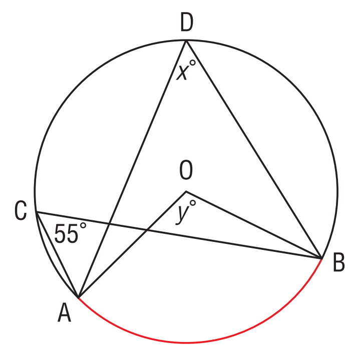
Since $\angle$ADB and $\angle$ACB are inscribed angles subtended by the same arc AB, these angles are congruent.

So, $x° = 55°$

Both the central $\angle$AOB and the inscribed $\angle$ACB are subtended by minor arc AB.
So, the central angle is twice the inscribed angle.

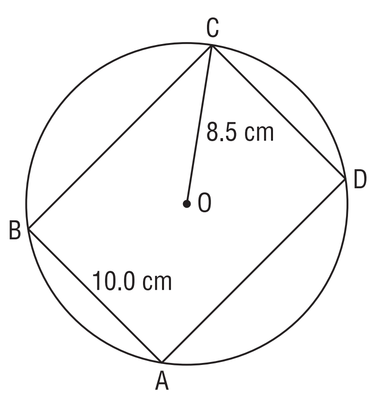
That is, $\angle AOB = 2\angle ACB$

$$y° = 2 \times 55°$$
$$= 110°$$



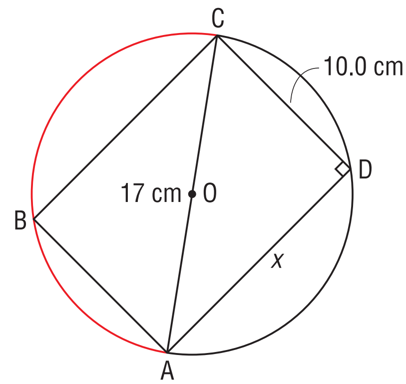
Example 2 Applying the Property of an Angle Inscribed in a Semicircle

Rectangle ABCD has its vertices on a circle with radius 8.5 cm.
The width of the rectangle is 10.0 cm. What is its length?
Give the answer to the nearest tenth.



A Solution

The length of the rectangle is AD.
Each angle of the rectangle is 90°.
So, each angle is subtended by a semicircle:
$\angle$ADC is subtended by semicircle ABC.
This means that each diagonal of the rectangle is a diameter of the circle:
AC is a diameter.



The radius of the circle is 8.5 cm,
so AC = 2×8.5 cm, or 17 cm.

Use the Pythagorean Theorem in right ΔADC to calculate AD.
Let the length of AD be represented by x.

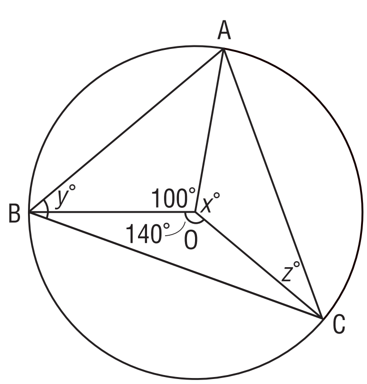
$$x^2 + CD^2 = AC^2$$
$$x^2 + 10^2 = 17^2$$
$$x^2 + 100 = 289$$
$$x^2 = 289 - 100$$
$$x^2 = 189$$
$$x = \sqrt{189}$$
$$\doteq 13.748$$

The rectangle is about 13.7 cm long.

A polygon whose vertices lie on a circle is an **inscribed polygon**.
In *Example 2*, rectangle ABCD is an *inscribed rectangle*.
Rectangle ABCD is *inscribed* in the circle.

Example 3 Determining Angles in an Inscribed Triangle

Triangle ABC is inscribed in a circle, centre O.
$\angle$AOB = 100° and $\angle$COB = 140°
Determine the values of x°, y°, and z°.

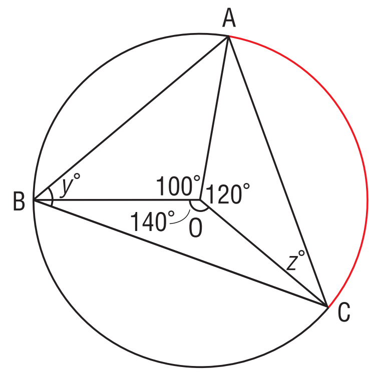


A Solution

The sum of the central angles in a circle is 360°.

So, $100° + 140° + x° = 360°$

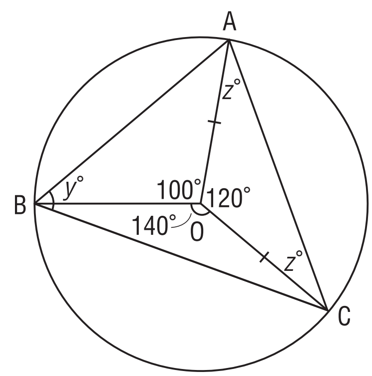
$$240° + x° = 360°$$
$$x° = 360° - 240°$$
$$= 120°$$



$\angle ABC$ is an inscribed angle and $\angle AOC$ is a central angle subtended by the same arc.

So, $\angle ABC = \frac{1}{2}\angle AOC$

$$y° = \frac{1}{2} \times 120°$$
$$= 60°$$



OA and OC are radii, so ΔOAC is isosceles, with $\angle OAC = \angle OCA = z°$

The sum of the angles in a triangle is 180°.

So, in ΔOAC,

$$120° + z° + z° = 180°$$
$$120° + 2z° = 180°$$
$$2z° = 180° - 120°$$
$$= 60°$$
$$z° = \frac{60°}{2}$$
$$= 30°$$

Discuss the ideas

1. How can the circle properties in this lesson help you decide where soccer players need to stand to have the same shooting angle on goal?

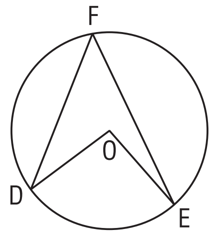
2. Suppose a circle has an inscribed angle. How do you identify the arc that subtends the angle?

Practice

Check

3. In each circle, identify an inscribed angle and the central angle subtended by the same arc.

a)



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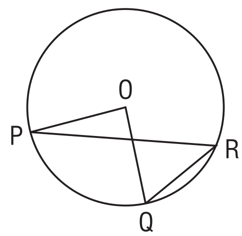

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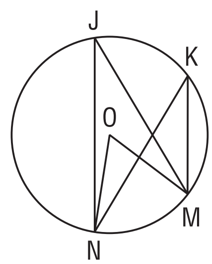
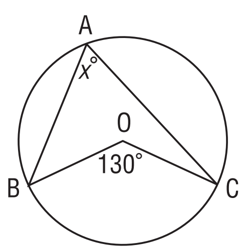


4. Point O is the centre of each circle. Determine each value of $x°$.

a)



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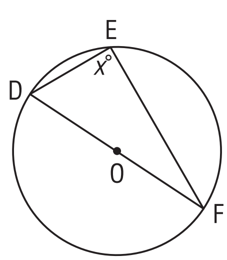

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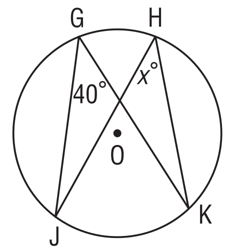

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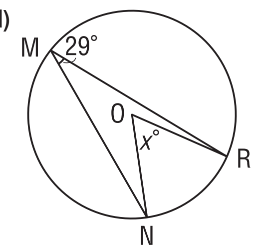
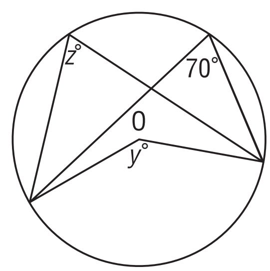


Apply

5. Point O is the centre of each circle. Label each vertex. Determine each value of $y°$ and $z°$. Which circle properties did you use?

a)



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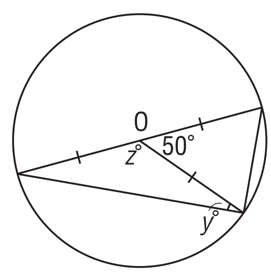
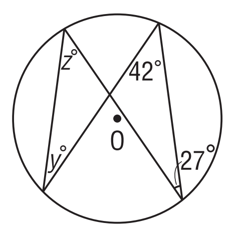
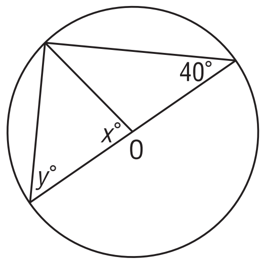


c)



6. Point O is the centre of each circle. Label each vertex. Determine each value of $x°$ and $y°$. Which circle properties did you use?

a)



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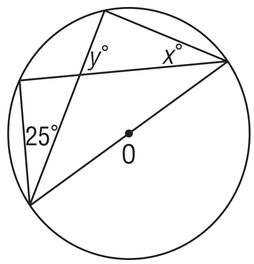


7. Construct a circle and two diameters PR and QS. Join the endpoints of the diameters to form quadrilateral PQRS.

a) What type of quadrilateral is PQRS? Use what you have learned in this lesson to justify your answer.

b) What type of quadrilateral is PQRS when the diagonals are perpendicular? Construct a diagram to check your answer.

8. Draw and label a diagram to illustrate:

a) the measure of the central angle in a circle is equal to twice the measure of an inscribed angle subtended by the same arc

b) the inscribed angles subtended by the same arc of a circle are equal

9. Rectangle PQRS is inscribed in a circle with radius 7 cm. The length of the rectangle is 12 cm.

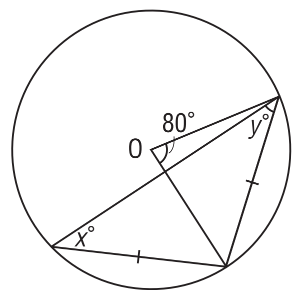
a) Sketch a diagram.

b) What is the width of the rectangle? Give the answer to the nearest tenth. Justify your solution.

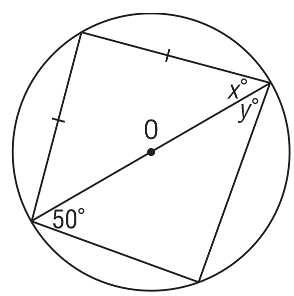
10. **Assessment Focus** Geometry sets often include *set squares*. A set square is a plastic right triangle. Trace around a circular object. Explain how you can use a set square and what you know about the angle in a semicircle to locate the centre of the circle. Justify your solution.

11. Point O is the centre of each circle. Label each vertex. Determine each value of $x°$ and $y°$. Which circle properties does each question illustrate?

a)



b)



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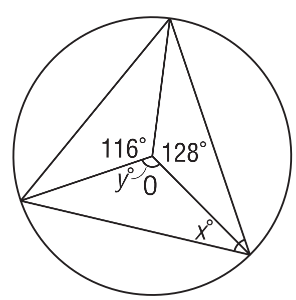
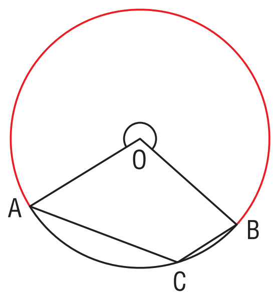


12. In *Investigate* on page 404, point C was on the major arc AB of a circle, centre O. Suppose C was on the minor arc AB. Do the circle properties that relate inscribed angles and central angles still apply? Investigate to find out. Justify your answer.



13. Some hockey players are approaching the goal. Two of them are the same distance from the end boards. Rana's shooting angle is 30° while Raji's is 35°.

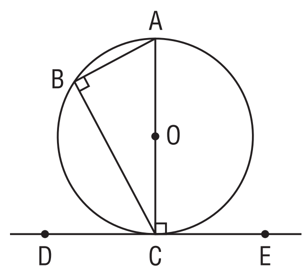
a) Sketch a diagram.

b) Who is closer to the middle of the ice? Explain your reasoning.

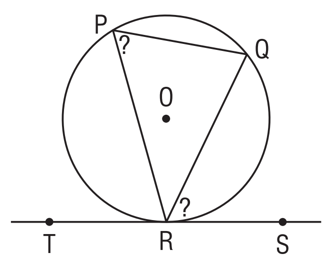
Take It Further

14. The *Seven Counters* game board on page 402 is an 8-pointed star inscribed in a circle. The vertices are equally spaced around the circle. What is the measure of the inscribed angle at each vertex of the star? Justify your solution.

15. The measure of ∠ACE between a tangent DE and the diameter AC at the point of tangency C is 90°. The measure of ∠ABC inscribed in a semicircle is also 90°.



a) How does the angle between a tangent and a chord appear to be related to the inscribed angle on the opposite side of the chord? That is, how is ∠QRS related to ∠QPR? Are ∠PRT and ∠PQR related in a similar way? Explain your reasoning.



b) Construct and measure accurate diagrams to verify the relationship in part a.

Reflect

Make a poster that summarizes the properties of angles in a circle.

Technology

Verifying the Angle Properties

FOCUS

- Use dynamic geometry software to verify the properties of angles in a circle.

Dynamic geometry software on a computer or a graphing calculator can be used to verify the circle properties in Lesson 8.3.

The diagrams show what you might see as you conduct the investigations that follow.

To verify the property of inscribed and central angles

1. Construct a circle.

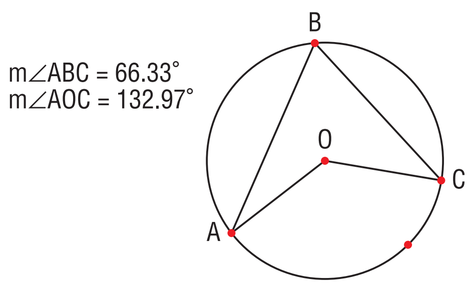
2. Mark three points on the circle.
Label them A, B, and C.
Label the centre of the circle O.

3. Join AB and BC. Join OA and OC.

4. Measure $\angle$ABC and $\angle$AOC.
What do you notice?

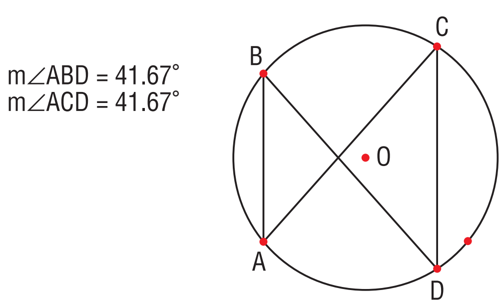
5. Drag point C around the circle. Do *not* drag it between points A and B.
Does the measure of $\angle$ABC change?
What property does this verify?

6. Drag point A or B around the circle. What do you notice about the angle measure relationship?



To verify the property of inscribed angles subtended by the same arc

1. Construct a circle.
2. Mark four points on the circle. Label them A, B, C, D in order. Label the centre of the circle O.
3. Join AB, AC, BD, and CD.
4. Measure ∠ABD and ∠ACD. What do you notice?
5. Drag point C around the circle. What do you notice about the angle measures? What property does this verify?



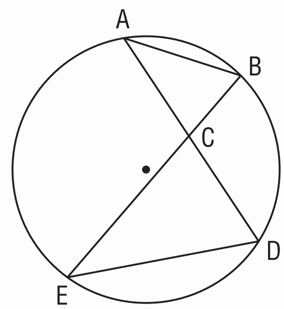
Check

1. In the first investigation, you dragged point C around the major arc AB. Predict what would happen if you dragged C to the minor arc AB. Use the software to confirm your prediction.
2. Use the software to confirm that all right triangles can be inscribed in a circle. Justify your strategy.

Start Where You Are

How Do I Best Learn Math?

Suppose I have to investigate two triangles like these in a circle.



➤ I could work alone or with others.

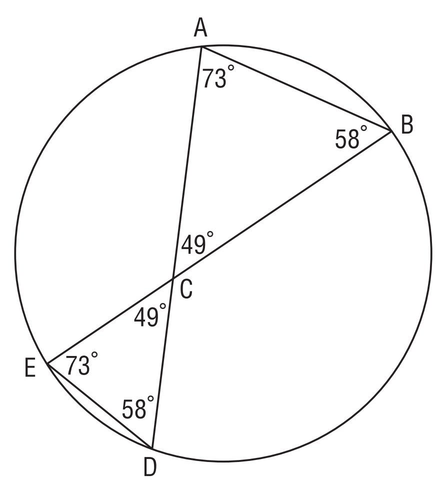
- Keena says, "I prefer to think things through on my own."
- Jetta says, "I like to discuss my ideas with a partner."
- Tyrell says, "I like to work in a group to get lots of ideas."

➤ I use what I know about angles.

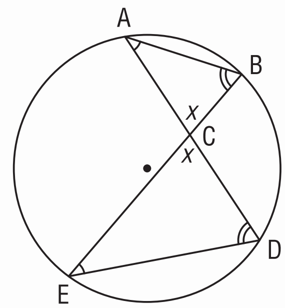
- Keena's method:
  I drew and labelled a diagram like the one above, then measured the angles in the triangles.
  I made sure that the diagram is big enough to be able to measure the angles with a protractor.
  I recorded the angle measures on the diagram.
  I noticed that pairs of angles in the two triangles are equal:

  $\angle ABE = \angle ADE = 58°$
  $\angle BAD = \angle BED = 73°$
  $\angle ACB = \angle ECD = 49°$

  So, the triangles are similar.
  $\Delta ABC \sim \Delta EDC$



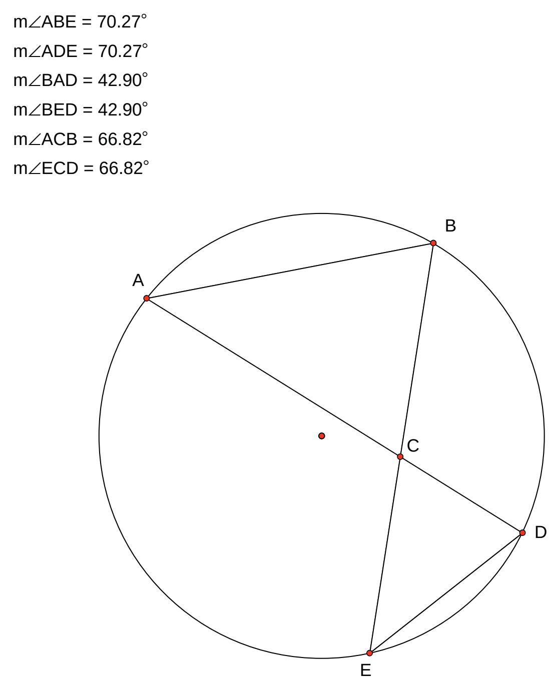
- Jetta's method:
  I reasoned from what I have learned about angles in a triangle.
  Arc AE subtends inscribed angles at B and at D.
  So, $\angle ABE = \angle ADE$
  Arc BD subtends inscribed angles at A and at E.
  So, $\angle BAD = \angle BED$
  Since two pairs of angles in the two triangles are equal, the angles in the third pair must also be equal, because the sum of the angles in any triangle is 180°.
  So, $\angle ACB = \angle ECD$
  Since 3 pairs of corresponding angles in two triangles are equal, the triangles are similar.
  $\Delta ABC \sim \Delta EDC$



- Tyrell's method:
  I used geometry software.
  I drew a circle and two intersecting chords.
  I then joined the ends of the chords to form two triangles.
  I labelled the vertices of the triangles.
  I used the software to measure the angles.
  I rounded the angle measures shown on the screen to the nearest degree.

  From the screen, I noticed that these angles are equal:
  $\angle ABE = \angle ADE = 70°$
  $\angle BAD = \angle BED = 43°$
  $\angle ACB = \angle ECD = 67°$

  Since 3 pairs of corresponding angles in two triangles are equal, the triangles are similar.
  $\Delta ABC \sim \Delta EDC$

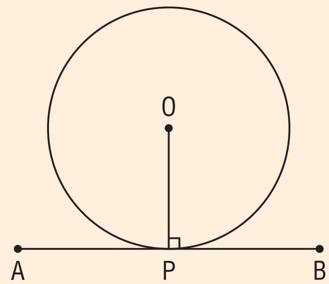


Check

1. Choose the way you best learn math.
   Investigate whether all rectangles can be inscribed in a circle.

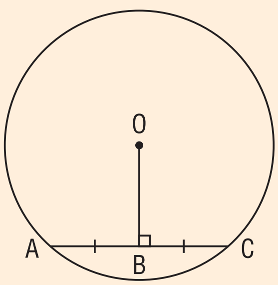
Study Guide

- A tangent to a circle is perpendicular to the radius at the point of tangency.
  That is, ∠APO = ∠BPO = 90°

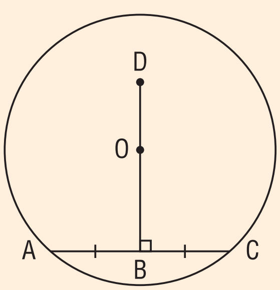


- The perpendicular from the centre of a circle to a chord bisects the chord.
  When ∠OBC = ∠OBA = 90°, then AB = BC

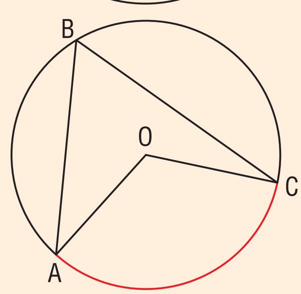
- A line segment that joins the centre of a circle to the midpoint of a chord is perpendicular to the chord.
  When O is the centre of a circle and AB = BC, then ∠OBC = ∠OBA = 90°



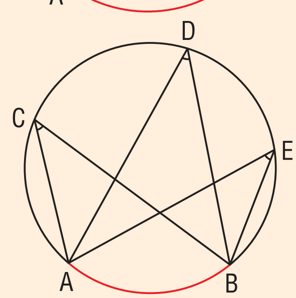
- The perpendicular bisector of a chord in a circle passes through the centre of the circle.
  When ∠OBC = ∠OBA = 90°, and AB = BC, then the centre O of the circle lies on DB.



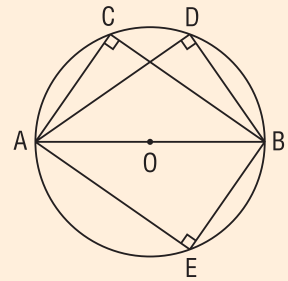
- The measure of a central angle subtended by an arc is twice the measure of an inscribed angle subtended by the same arc.
  ∠AOC = 2∠ABC, or
  ∠ABC = $\frac{1}{2}$ ∠AOC



- All inscribed angles subtended by same arc are congruent.
  ∠ACB = ∠ADB = ∠AEB



- All inscribed angles subtended by a semicircle are right angles.
$\angle ACB = \angle ADB = \angle AEB = 90°$

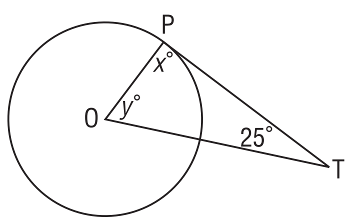


Review

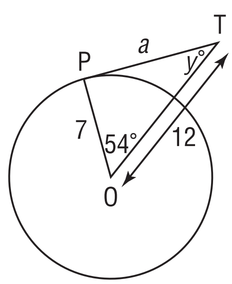
Give the answers to the nearest tenth where necessary.

8.1 **1.** Point O is the centre of each circle. Segments PT and QT are tangents. Determine each value of $x°$, $y°$, a, and b. Show your work.

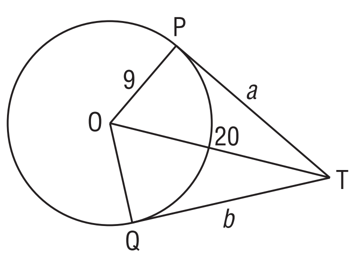
a)



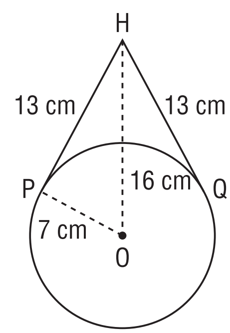
b)



c)

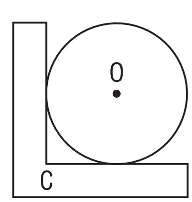


2. A circular mirror is suspended by a wire from a hook, H. Point O is the centre of the circle and is 16 cm below H. Explain how you know that the wire is *not* a tangent to the circle at P and at Q.



3. Draw a circle with centre O. Mark a point P on the circle. Explain how to draw a tangent to the circle. Which circle property did you use?

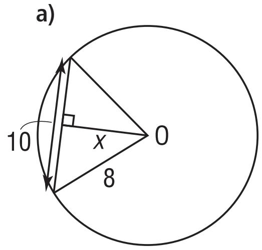
4. A circular plate is supported so it touches two sides of a shelf. The diameter of the plate is 20 cm. How far is the centre O of the plate from the inside corner C of the shelf? Which circle properties helped you find out?



8.2

5. Point O is the centre of each circle.
Determine each value of x.
Justify your answers.

a)



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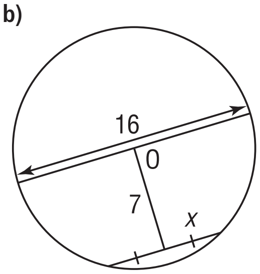


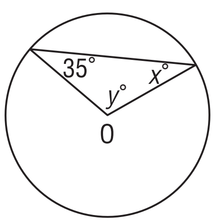
6. A dream catcher with diameter 22 cm is strung with a web of straight chords. One of these chords is 18 cm long.

a) Sketch a diagram.

b) How far is the chord from the centre of the circle? Justify your solution strategy.

7. Point O is the centre of each circle.
Determine each value of $x°$ and $y°$.
Which circle properties did you use?

a)



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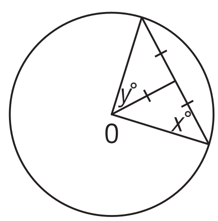
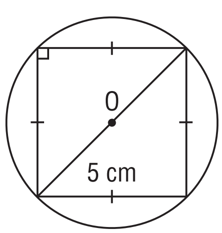


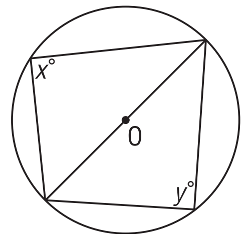
8. A square has side length 5 cm. It is inscribed in a circle, centre O.
What is the length of the radius of the circle? How do you know?



8.3

9. Point O is the centre of each circle.
Determine each value of $x°$ and $y°$.
Justify your answers.

a)



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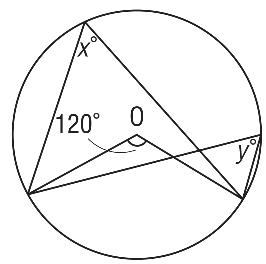

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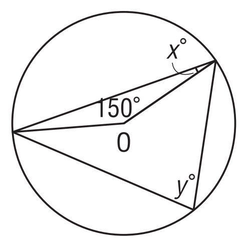
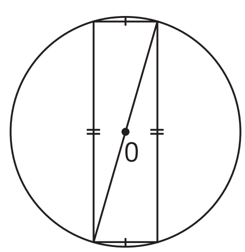
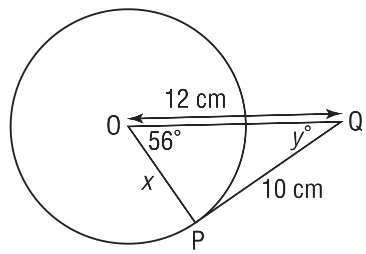


10. A rectangle is inscribed in a circle, centre O and diameter 36.0 cm. A shorter side of the rectangle is 10.0 cm long. What is the length of a longer side? How do you know?

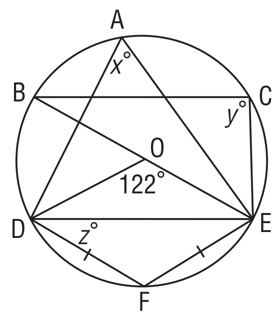


Practice Test

1. Point O is the centre of the circle.
Point P is a point of tangency.
Determine the values of x and $y°$.
Give reasons for your answers.



2. Point O is the centre of the circle.
Determine the values of $x°$, $y°$, and $z°$.
Which circle properties did you use each time?



3. A circle has diameter 6.0 cm. Chord AB is 2.0 cm from the centre of the circle.

a) Sketch a diagram.

b) How long is the chord AB?

c) Another chord, CD, in the circle is 2.5 cm from the centre of the circle.
Is chord CD longer or shorter than chord AB? Justify your answer.

4. Use what you know about inscribed and central angles to explain why the angle inscribed in a semicircle is 90°.

5. Where is the longest chord in any circle? How do you know? Draw a diagram to illustrate your answer.

6. A circle has diameter 16 cm.

a) Which of the following measures could be distances of chords from the centre of this circle? How could you check your answers?

i) 4 cm **ii)** 6 cm **iii)** 8 cm **iv)** 10 cm

b) For each possible distance you identified in part a, determine the length of the chord.

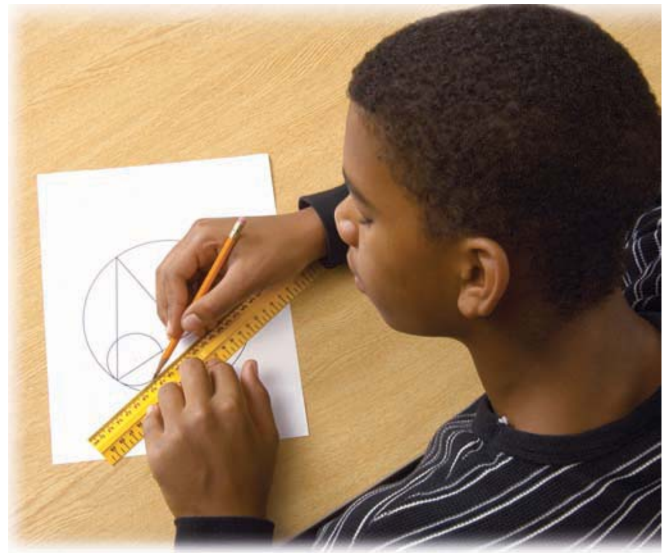
7. a) Construct a circle and mark points P and Q to form major and minor arcs PQ.

b) Construct inscribed ∠PRQ subtended by minor arc PQ.

c) Construct inscribed ∠PSQ subtended by major arc PQ.

d) How are ∠PRQ and ∠PSQ related? Justify your answer.

Unit Problem — Circle Designs



Many works of art, designs, and objects in nature are based on circles.
Work with a partner to generate a design for a corporate or team logo.

Part 1
Sketch a design that uses circles, tangents, and chords.
Use your imagination to relate circles to a business or sports team.

Part 2
Work with geometry tools or computer software to draw your design.
Measure and label all angles and lengths that demonstrate the circle properties.
Some lines or features at this stage may disappear or be covered by the final coloured copy. So, ensure you have a detailed design copy to submit that demonstrates your understanding of the geometry.

Part 3
Produce a final copy of your design. You may cover or alter the underlying geometry features at this point if it enhances your design.

Your work should show:

- sketches of your design
- a detailed, labelled copy of your design that shows circle geometry properties
- written explanations of the circle properties you used in your design
- a final coloured copy of your design with an explanation of its purpose, if necessary

Reflect on Your Learning

Explain how knowing the circle properties from this unit can help you determine measurements of lengths and angles in circles.

UNIT 9

Probability and Statistics

Are you an average teenager? How could you find out how your answers to these questions compare with those of the average Canadian teenager?

What You'll Learn

- Understand the role of probability in society.
- Identify and address problems related to data collection.
- Use either a population or a sample to answer a question.
- Develop and implement a plan to collect, display, and analyze data.

Why It's Important

People make decisions based on data. People can make informed decisions if they understand what the data really mean and where they come from, and are confident the data are accurate.

Key Words

- population
- census
- sample
- valid conclusions
- rubric

9.1 Probability in Society

FOCUS

- Explain how probability is used outside the classroom.

This season, Haley made 44 out of 50 basketball shots she attempted.
What is the probability Haley will sink the next shot?
What assumptions do you make?

Investigate

2

From the statements below, identify the different probabilities that Jean-Guy considers in a day. List an assumption associated with each statement. Explain how the situation would change if that assumption were not true.

- Jean-Guy noticed that, in the last month, 70% of the time the bus was 3 minutes late. So, he takes his time with breakfast today.
- Lately, Jean-Guy's math teacher checks homework 4 days a week. So, Jean-Guy makes sure he has time to complete his homework today.
- At school, Jean-Guy and his friends agree that their lacrosse team has a 95% chance of making the finals.
- In health class, Jean-Guy's teacher reads a magazine that claims 172 out of 1000 male smokers develop lung cancer, but only 13 out of 1000 males who do not smoke develop lung cancer.

Reflect & Share

Share your assumptions with those of another pair of students.
Discuss which assumptions you think are the most likely and least likely to be true.
Which probability could have been most influenced by personal opinion?
How do you think the other probabilities were determined? Explain.

Connect

Probability refers to the likelihood that an event will occur.

By collecting and analyzing data, predictions can be made about the likelihood that a certain event will occur. For example, meteorologists study past weather data to make predictions about future conditions. A 40% probability of snow means that under similar conditions in the past it snowed 40% of the time, or 4 times out of 10.

A probability of 40% can be expressed as 4 out of 10, $\frac{4}{10}$, or 0.4.

When you flip a coin, there is a 0.5 probability the coin will land heads up. Despite this probability, you may feel strongly that the coin will land tails up. In this case, you have made a *subjective judgment*.

Example 1 Identifying Decisions Based on Probabilities and Judgments

Explain how each decision is based on theoretical probability, experimental probability, subjective judgment, or any combination of these.

a) It is Ausma's experience that 4 out of 5 times the prize in the cereal box is found at the bottom of the box. So, Ausma opens the bottom of the cereal box to find her prize.

b) Two friends are rolling a die. Out of eight rolls made, a "4" came up 7 times. Amith predicts the next roll will likely not be a "4," since each number has an equal chance of being rolled. Maria decides the die is unfair since 7 out of 8 rolls revealed a "4."

A Solution

a) Ausma's decision to open the bottom of the cereal box to find her prize is based on past experience. This is an example of experimental probability.

b) Amith's decision that the next roll will likely not be a "4" is based on theoretical probability. Amith knows that the probability of rolling each number is 1 out of 6, so the probability of rolling any number other than a "4" is 5 out of 6.
Based on previous rolls, Maria has noticed that the experimental probability of rolling a "4" is 7 out of 8. She knows that each number should have an equal probability of being rolled. So, she makes a subjective judgment that the die must be unfair.

Often, we make predictions about an outcome based on assumptions associated with a given probability.
If these assumptions change, the prediction may not match the outcome.

Example 2 Explaining How Assumptions Affect a Probability

In past baseball games, Alice made 2 hits for every 5 times she went up to bat.

a) In the next game, suppose Alice goes up to bat. What is the probability that she will get a hit? What assumptions are you making?

b) For each assumption, explain how the predicted outcomes might change if the assumption changes.

A Solution

a) The experimental probability of Alice hitting the ball is 2 out of 5, or 40%. We assume that the next team Alice plays against is at the same level of ability as previous teams she has played.

b) If the opposing team is more able, then Alice will probably make fewer hits. The likelihood that Alice has a hit would be less than 2 out of 5. If the opposing team is less able, then Alice will probably have more than 2 hits. In this case, the likelihood that Alice has a hit would be greater than 2 out of 5.

Sometimes a probability can be used to support opposing views.

Example 3 Using a Probability to Support Opposing Views

Jon wants to learn how to snowboard but does not want to take lessons. His mother insists that Jon take lessons. Jon and his mother find an article that claims:


68% of snowboarding injuries occur during beginner lessons

Explain how both Jon and his mother can use this statistic to support their opinions.

A Solution

68% of snowboarding injuries occur during beginner lessons. Jon's mother can argue that this statistic proves lessons are important because beginners are prone to accidents. Jon can argue that this statistic is a good reason not to take lessons because the likelihood of getting injured during the lesson is high.

Discuss the ideas

1. When you toss a coin, what assumptions are you making when you say the probability of it landing heads up is $\frac{1}{2}$?

2. Car insurance for teenagers is more expensive than for adults because the probability of an accident is greater for teenage drivers. What assumptions is an insurance company making when it charges a teenage driver more for insurance?

Practice

Check

3. Indicate whether each decision is based on theoretical probability, experimental probability, or subjective judgment. Explain how you know.
   a) The last two times Andrei won a prize at a coffee shop, he ordered a medium hot chocolate. Andrei never won when he ordered a large hot chocolate, so today he orders a medium hot chocolate.
   b) Instead of buying her own lottery ticket, Martha pools her money with the people at work to buy more tickets and increase her chances of winning.
   c) Anita boards the last car of a train because, in the past, the last car always had available seats.
   d) Doug will not travel by airplane even though experts say it is safer to fly than drive.

4. What assumptions is each person making?
   a) Based on past math quizzes, Claudia says she has a 90% chance of getting a perfect score on her next math quiz.
   b) Six times out of ten, Omar gets stuck in traffic when he leaves work. So, he calculates that his chances for getting stuck in traffic today after work are 60%.

Apply

5. The weather forecast is 70% chance of rain. Winona had planned to go canoeing. Explain how the decision she makes may be based either on probability or on subjective judgment.

6. The student council has a draw for a prize during the school dance. Lei decides not to enter the draw because all of his classmates have entered and he feels unlucky today. Is Lei's decision based on probability, on subjective judgment, or both? Explain.

7. One year, the probability of *not* recovering a stolen vehicle in Montreal was 44%. How could politicians use this fact to argue that:
   a) more money should be allotted to searching for stolen vehicles
   b) more money should not be allotted, and instead should go to different causes

8. Vanessa observes her birdfeeder at the same time each day for a week. She notes that 32 of the 100 birds which visit the feeder are cardinals. She concludes that, in general, there is a 32% probability a bird visiting the feeder will be a cardinal.
   a) What assumptions did Vanessa make?
   b) If each assumption changes, how might the predicted outcome change?

9. Kathryn read this headline:


Poll reveals 30% support Bradford, 70% support Choo in next election

Kathryn says that if she polled the next 10 people she passed on the street, 7 of them would be voting for Choo.
   a) What assumption is Kathryn making?
   b) Explain what the effect might be if the assumption were not true.

10. A DNA match was found between a blood sample and a suspect. A forensic scientist testifies that there is a 1 in 7000 chance the blood sample is from someone other than the suspect. Describe how two lawyers could use this statistic to support different positions.

11. **Assessment Focus** An advertisement for acne treatment boasts:

   a) Explain how a teenager's decision on whether to try this acne treatment could be based on probability and subjective judgment.
   b) If the teenager does decide to try the acne treatment, what assumptions might he be making? For each assumption, explain how the predicted outcome of the treatment might change if the assumption changes.

12. a) Provide 2 examples of how statistics are used in the media to sell a product.
   b) Why do advertisers use numbers in these ads? Do you think using statistical data makes the ads more effective?
   c) For each example, list some assumptions associated with the statistic.

13. Look at newspapers, magazines, or on the internet. Give 2 examples of how politicians or environmentalists use probability.

Take It Further

14. Shaquille O'Neal's free throw percentage during one season was 62%. A teacher points out that this means each time Shaquille attempted a free throw during that season, his probability of making the shot was 62%.
A student then says:
"Shaquille either makes the shot or he doesn't. So, isn't the probability 50%?"
Explain the flaw in the student's thinking.

15. Research 2 occupations that use probability. Explain the role of probability in each occupation.

16. The annual Farmer's Almanac makes predictions about long range weather patterns. Investigate to find out how these predictions are made. What assumptions is the almanac making?

17. According to Transport Canada, in 2004, there were 34 fatalities due to air travel and 2730 fatalities due to road travel.

a) What impression does this information give? How might this information be misleading?

b) What additional information would you need to determine whether travelling by plane or by car is safer?

Reflect

Think of 2 decisions that could be influenced by probabilities.
What assumptions would you be making about each probability?
How might the probabilities be different if the assumptions were not true?

Math Link

Your World

Probabilities are used in risk assessment. To compare the safety of certain sport utility vehicles (SUVs) and minivans, researchers subjected both to crash tests while the vehicles moved at 56 km/h. Here are the results:

Probability of:	SUV	Minivan
life threatening head injury	16%	2%
life threatening chest injury	20%	4%
life threatening leg injury	35%	1%

How might insurance companies use this information?
How might car manufacturers use this information?

GAME

Cube Master

How to Play

1. Choose a “dealer”. The dealer selects any 30 cubes. (For example, the dealer could choose 5 red cubes, 12 green cubes, and 13 blue cubes.)
   No other player should know how many cubes of each colour were selected.
   The dealer places the 30 cubes in a container.
2. Each player records a guess of how many cubes of each colour are in the container.
   Players should not share these guesses.
3. The players take turns selecting one cube from the container, then returning the cube.
   (The dealer makes sure the players cannot see what is in the container.)
   Players note which colour was selected each time.
   Stop after 10 cubes have been selected and returned.
4. Players now adjust their initial guesses by considering the colours of the cubes selected.
5. Repeat Steps 3 and 4 two more times.
6. Players compare their final estimates with the actual numbers of cubes to calculate their points.
   The player with the fewest points wins.

For example:

	Actual Number of Cubes	Player’s Final Estimate	Number of Points (Difference between actual number and estimate)
Red	5	4	1
Green	12	11	1
Blue	13	15	2
Total Number of Points:			4

7. Repeat the game until everyone has had the opportunity to be the dealer.

Share your strategies with the other players.
Whose strategy worked best? Why?

You will need

- more than 30 cubes (blue, green, and red)
- a container for the cubes

Number of Players

- 4

Goal of the Game

- Use experimental probability and subjective judgment to estimate the numbers of coloured cubes in a container.

9.2 Potential Problems with Collecting Data

FOCUS

- Describe how the collection of data may be adversely affected.

Suppose your friend asks you, "Who do you think will win the Western Hockey League (WHL) playoffs?"

What might affect your answer to this question?

Investigate

Choose one person in the group to be the leader.
The leader will read the questions below, while the others write down their answers.
The leader should ask the questions exactly as they are written, without explanation.

1. How large is this textbook? (Point to this textbook when asking the question.)
2. Do you think students are working too much because of the large amount of homework assigned?
3. What is your favourite genre of music? Choose one from: Rock, Pop, Country

Share all the answers with the group.
Did you interpret the questions the same way as your group? Why?
Describe any problems with each question.
Rewrite each question to avoid these problems.

Reflect & Share

Share your rewritten questions with another group.
Discuss how the rewritten questions avoid potential problems.
How might a poorly worded question affect data collection?
What sort of wording should someone collecting data use?
What sort of wording should that person avoid?

Connect

There are several factors that might lead to problems with data collection.

Potential Problem	What It Means	Example
Bias	The question influences responses in favour of, or against the topic of the data collection.	Suppose a person asks: *Don't you think the price of a movie ticket is too high?* This person has a bias against the current ticket price, and the bias influences how the survey question is written.
Use of Language	The use of language in a question could lead people to give a particular answer.	If you ask: *Don't you think the price of a movie ticket is too high?*, the question may lead people to answer yes. A better question would be: *Do you think the price of a movie ticket is too high, too low, or fair?*
Timing	*When* the data are collected could lead to particular results.	A survey is conducted to find opinions on the need for a vehicle to have winter tires. The results may be different if the survey is conducted in August instead of February.
Privacy	If the topic of the data collection is personal, a person may not want to participate or may give an untrue answer on purpose. Anonymous surveys may help.	People may not want to participate in a study on weight if it means stepping on a scale in front of other people.
Cultural Sensitivity	Cultural sensitivity means that you are aware of other cultures. You must avoid being offensive and asking questions that do not apply to that culture.	Suppose you wanted to know the favourite method of cooking ham, and you asked: *Please circle your favourite method:* *BBQ* *Bake* *Fry* This question does not apply to everyone because many people do not eat ham. A better question would be: *If you eat ham, name your favourite method of cooking it.*
Ethics	Ethics dictate that collected data must not be used for purposes other than those told to the participants. Otherwise, your actions are considered unethical.	Suppose you tell your classmates that you want to know their favourite snacks to help you plan your birthday party. If you then use the information to try to sell your classmates their favourite snacks between classes, your actions would be unethical.
Cost	The cost of collecting data must be taken into account.	If you need to pay for printing the questionnaires, or to pay people to collect the data, the cost may be more than you can afford.
Time	The time needed for collecting the data must be considered.	A survey that takes an hour to complete may be too long for most people. This would limit the number of people willing to participate.

Example 1 Identifying and Eliminating Potential Problems

For each survey question, explain why a problem may occur and the effect it would have on the results. Suggest how each problem could be avoided.

a) A survey is conducted to find out if citizens think the local government should provide more money for youth activities. The question asked was: "Would you support an increase in taxes to create more skate parks?"

b) A survey is conducted to find out the level of school spirit. Students are polled about their level of school spirit after the soccer team wins the championship.

A Solution

a) The use of language in the survey question could be problematic.
The question emphasizes what citizens would lose; that is, their taxes would increase. The question also downplays what citizens would gain by only mentioning skate parks, instead of a variety of activities.
Most people would probably respond by saying they would not support an increase in taxes to build more skate parks.
A better question would be: "Do you think the local government should supply more funds for youth recreational activities?"

b) The timing of the survey question could be problematic.
Since the school's soccer team just won the championship, the level of school spirit would be higher than usual.
The results of the survey may show a higher level of school spirit than if the survey was conducted at another time.
Asking students the same question a month later, when no school event is occurring, should produce more accurate results.

Example 2 Analyzing Data Collection for Problems

Kublu and Irniq plan to open a shop in Saskatoon that would sell traditional Inuit crafts.
To ensure Saskatoon is the best place for their business, they want to survey residents to find out how popular Inuit crafts are.
Kublu knows that they would get the most accurate results if each household in Saskatoon is surveyed, but Irniq points out that this is problematic.
Explain why.

A Solution

The number of households in Saskatoon is great. Kublu and Irniq may have problems related to cost and time.
The cost of printing and mailing enough surveys for each household would be very high. Also, Kublu and Irniq should provide an envelope and stamp for each household to return the survey. This would be an additional cost.
The time it would take to print, mail, and collect the surveys for all the households would be too long.

Example 3 Overcoming Potential Problems of Data Collection

Antonia wants to find out if there is a relationship between household income and how much people spent on Christmas presents.
Identify potential problems Antonia may encounter, and explain how she could deal with the problems.

A Solution

Christmas is not celebrated by all cultures, and so the survey question does not apply to everyone. An appropriate opening question for the survey might be: "Do you celebrate Christmas?" If a person responds "No", then he or she will not need to answer the other question in your survey.

Information about income and spending habits is personal, so people may be uncomfortable revealing it. An anonymous survey would be appropriate.

The use of language may influence responses.
Examples of inappropriate or intrusive questions would be:
"How much do you make?" and "How much do you spend?"
A better question might be:
"Is the amount you spend on Christmas presents:
– greater than your weekly income?
– less than your weekly income?
– equal to your weekly income?"

Discuss the ideas

1. How could the use of language affect the data collected? Give an example.

2. Two factors that may influence data collection are *time* and *timing*. What is the difference between these?

Practice

Check

3. Name a problem with each data collection.

a) After the first week of school, your principal asks you and your friends how you are enjoying school.

b) An online magazine asks readers either to agree or disagree with the statement: "If you find a $20-bill, you turn it in."

c) Brenda asks her classmates if they think girls should not be allowed to cover their heads in school.

d) To discover the most popular kind of movie at his school, Carlos plans to ask each student what her or his favourite kind of movie is.

Apply

4. For each scenario in question 3:

a) Describe the effect each problem would have on the data collection.

b) How could each problem be overcome?

5. Parinder wanted to find out how often the computers in her school were being used. She asked students the question: "How much time do you spend on the computer each week?"

a) How do you think her schoolmates will interpret this question?

b) How could the question be rewritten so it would more accurately reflect what Parinder wants to know?

c) Who might be interested in her findings? Why?

6. Andrew went to each class in his school and asked for a show of hands to find out how many students had ever been bullied at school. Only 2 students raised their hands. Andrew concluded that bullying was not a major problem at his school.

a) Is this a reasonable conclusion? Explain.

b) Describe a better method for conducting Andrew's survey.

7. Trinity wants to find out how football fans feel about building a new indoor football stadium for a Canadian Football League team. She goes to the stadium to survey fans after a winning game on a warm August evening.

a) Describe how the timing of her question may influence the responses.

b) In what setting might the responses be different than those Trinity received?

8. a) Describe how each question reveals a bias of the questioner.

i) Do you think it is a good idea to use DNA tests to convict a violent criminal?

ii) Do you think gas guzzling SUVs should be banned?

iii) Do you think students should be allowed to use spell check because it automatically improves spelling?

b) Rewrite each question to eliminate the bias. Explain how your question is an improvement.

9. Rebecca was looking for a cell phone service provider. She surveyed her friends and asked who their service providers were. Based on these data, she chose the provider that her friends used more than any other.

a) Do you think Rebecca's question reflected what she wanted to know? Explain.

b) What questions might have helped Rebecca to make a more informed decision?

10. A fashion website is conducting a survey. Sasha answered questions about his favourite brands of clothing, then provided an email address as a login to the site in the future. Shortly after this, his inbox was full of emails advertising a new brand of clothing.

a) Which important factor did the survey designers overlook? How is this problematic?

b) How can the survey designers avoid this problem?

11. Provide an example of data collection where the cost and time needed to complete the collection may lead to problems.

12. **a)** Write 3 questions people would prefer to answer anonymously.

b) For each question, describe what the results might be if the participants were not anonymous.

13. **Assessment Focus** Bridget wants to find out how much the average grade 9 student spends on clothes each month.

a) Identify potential problems she may encounter related to 3 of these factors: use of language, ethics, cost, time, timing, privacy, cultural sensitivity

b) For each potential problem in part a, explain how Bridget could avoid the problem.

14. **a)** Describe 2 possible data collections that might be problematic because of the time of year they are conducted.

b) Suggest a better time that each should be conducted.

Take It Further

15. Common methods of surveying are by personal interviews, over the phone, or by email. Identify potential problems associated with each method of surveying.

16. **a)** Why might questions about Hanukkah be culturally sensitive?

b) Think of 3 more topics that might be culturally sensitive. Explain why.

c) Design a culturally sensitive survey question about one of the topics in part a or b. Explain how you would collect the data to address the cultural sensitivity.

Reflect

Why is it important to identify and overcome sources of potential problems in data collection?

9.3 Using Samples and Populations to Collect Data

FOCUS

- Select and defend the choice of using a population or a sample.

To estimate the number of salmon in a river, biologists use a strategy called *mark and recapture*. At one place in the river, biologists capture some fish. Each fish is marked with a tag, then released into the river. At a different place in the river, biologists recapture fish. They track the numbers of marked and unmarked fish caught. They can then estimate the salmon population.

Investigate

2

Each pair of students will need 50 small pieces of paper.

➤ Create a population of 50 fish by labelling each piece of paper with either "F" for female or "M" for male.

➤ Record the percents of female and male fish in your population.

➤ Fold the pieces of paper and place them in a box.

Trade populations with another pair of students.

➤ Choose a sample of 10 fish from the other pair's population. Record the numbers of male and female fish in your sample. Use this to estimate the percents of male and female fish in the population.

➤ Repeat the preceding step by choosing samples of 20 fish, and then 40 fish.

➤ Did your estimates of the percents of male and female fish change as the sample size changed? Explain.

Reflect & Share

Compare your estimates with the actual percents recorded by the other pair of students. How did your estimates compare with the actual percents? In general, how did the estimates change as the size of the sample increased?

Connect

When collecting data, the **population** is the group about which you are getting information.
A **census** is conducted when data are collected from each member of the population. For example, suppose you test game consoles made in a factory for defects, then *all* the game consoles made in the factory are the population. If you test each game console, then you have conducted a census.

A census can be costly, time consuming, and difficult or impossible to complete. So, a census is only used when an issue is important or when the population is small.

If a census is not feasible or necessary then data are collected from a small portion, or **sample**, of the population. When the sample chosen is representative of the population, the data collection provides **valid conclusions**. For example, testing 100 game consoles out of 1000 made each day is a sample. If those consoles tested represent the typical quality of consoles made in the factory, the conclusions of the data collection will be valid.

Care must be taken when determining the appropriate size of the sample. If the sample is large, the data collection could be costly or time consuming. If the sample is small, then it may not be representative of the population.

Example 1 Explaining Why Data Are Collected from Populations

In each case, explain why a population was surveyed instead of a sample.

a) To determine the average number of siblings of his classmates, Carlos surveyed each person in the class.

b) Every 5 years, Statistics Canada conducts a census. One question in the survey is used to determine the ages of the people in each household.

A Solution

a) Carlos knows that surveying the entire population will produce exact results, rather than estimates. So, he chose to survey the entire population, the whole class, because it would not take long or cost him anything.

b) A census was completed because of the importance of the question. The government requires data about the ages of Canadians so that it can budget for services such as day-care centres, schools, and senior citizens' homes.

Example 2 Reasoning Why and When Samples Should Be Used

The student leadership team is planning a school dance. To attract grade 9 students to the dance, the team decided to collect data about the preferred music of the grade 9 students. The team set up in the hallway to collect the data. By the end of the day it had surveyed 73% of the grade 9 students.

a) Why do you think the data were collected from a sample instead of the entire population?

b) Will the opinions of the sample likely reflect those of the population? Explain.

A Solution

a) There was probably not enough time or people available to ask all grade 9 students. It would also require a lot of effort to find each grade 9 student, especially with absences.

b) Since the majority of students, 73%, were asked, it is likely that their opinions will reflect those of the entire population.

Example 3 Identifying and Critiquing the Use of Samples

In each case, identify if data were collected from a sample or a population. Wherever a sample was used, explain if you think the conclusion would be valid.

a) A province considers banning cell phones in all of its schools. To determine the opinions of students on this issue, you poll each student in your school.

b) To determine which politician is expected to win the municipal election, every person over 18 and who is eligible to vote in the election is polled.

c) To determine the average lifetime of a type of light bulb, 150 light bulbs were selected randomly from the production line and tested.

A Solution

a) Sample: The population is all students of all schools in the province. By asking only the students in your school, your results are based on a sample. If the students in your school do not represent typical students in the province, the conclusion will not be valid. For example, if all students in your school own cell phones, your conclusion would probably be not to ban cell phones. However, not every student in the province owns a cell phone. So, your results would not be representative of the population.

b) Population: All possible voters are polled.

c) Sample: Since not all light bulbs were tested, the results are based on a sample. It would not make sense for the whole population to be tested, since all light bulbs would be destroyed in the process. There would be no light bulbs left to sell. Since a fairly large number of light bulbs were tested, the results will likely give a good estimate of the lifetime of a light bulb. So, the conclusion about the lifetime of a light bulb is likely valid.

Discuss the ideas

1. What factors do you need to consider when you collect data from either a population or a sample?

2. What does "valid conclusion" mean? Provide an example where the conclusion based on a sample is not valid.

Practice

Check

3. In each case, describe the population.
   a) The management team of a shopping mall in Comox wants to know how to attract more people between the ages of 13 and 25 to the mall.
   b) A juice company wants to determine the average volume of juice in a 1-L carton.
   c) A board of education wants to find out which schools need renovations.
   d) The government wants to determine the average age of First Nations people in Nunavut.

4. In each case, are the data collected from a census or a sample?
   a) To determine the favourite TV show of grade 9 students in a school, all grade 9 students in the school are surveyed.
   b) To find out if customers of a chain of coffee shops are happy with the service, some customers in every shop were surveyed.

Apply

5. Identify the population you would sample to find out opinions on:
   a) bus fares b) the GST c) cost of day care
   d) emergency room wait times

6. Courtney surveys her friends and finds that 68% of them have an MP3 player. She reports that 68% of the grade 9 students have an MP3 player. James surveys the entire grade 9 population and discovers that 51% have an MP3 player.
   a) Whose conclusion is more likely to be valid? Explain.
   b) Why might the other student's conclusion not be valid?

7. For each situation, explain why data are collected from a sample and not a census.

a) to determine the number of hours an AAA battery will last in a calculator

b) to determine the number of First Nations children in Canada who speak Cree

8. Should a census or sample be used to collect data about each topic? Explain your choice.

a) the effectiveness of a new suntan lotion

b) the popularity of a fruit-flavoured yogurt

c) the number of grade 9 students in your school with braces

d) the number of your friends who like to play computer games

9. In each case, do you think the conclusion is valid? Justify your answers.

a) Irina surveyed 20 students to find out if they eat breakfast. All the students said yes. Irina concluded that everyone in the school eats breakfast.

b) To test for pesticide pollution, a scientist collects and tests one vial of water from a river. From the results, a local newspaper reporter concludes that there are dangerous levels of pesticide in the river.

10. Assessment Focus Suppose you are the manager of a high school cafeteria. You want to create a new breakfast and lunch menu for the students.

a) What population are you interested in surveying?

b) Would you survey a sample or population? Explain.

c) If you had to use a sample, what would you do to make sure your conclusions are valid?

11. In each case, provide an example and justify your choice.

a) Collecting data from a population, rather than a sample, is more appropriate.

b) Collecting data from a sample, rather than the entire population, is more appropriate.

Take It Further

12. a) Describe a situation where a sample:

i) represents a population

ii) does not represent a population

b) What changes would you make to the sample in part ii so that the conclusions would be valid?

13. Every 5 years, the *Census of Agriculture* is sent to every farm household across Canada. This census collects data on topics such as crop area, livestock, farm labour, machinery, and expenses. Choose one of these topics. Explain why you think it is important enough for the government to conduct a census.

Reflect

Describe when to use a census and when to use a sample.

Using *Census at School*

FOCUS

- Search a Canadian database for information, then make comparisons.

Statistics Canada is a government agency that collects data on Canadian citizens. *Census at School* is an international online project that engages students from grades 4 to 12 in statistical enquiry. Data on students from 8 to 18 years old can be found at the *Census at School* website.

You can use *Census at School* to find data about Canadian youth under headings such as:

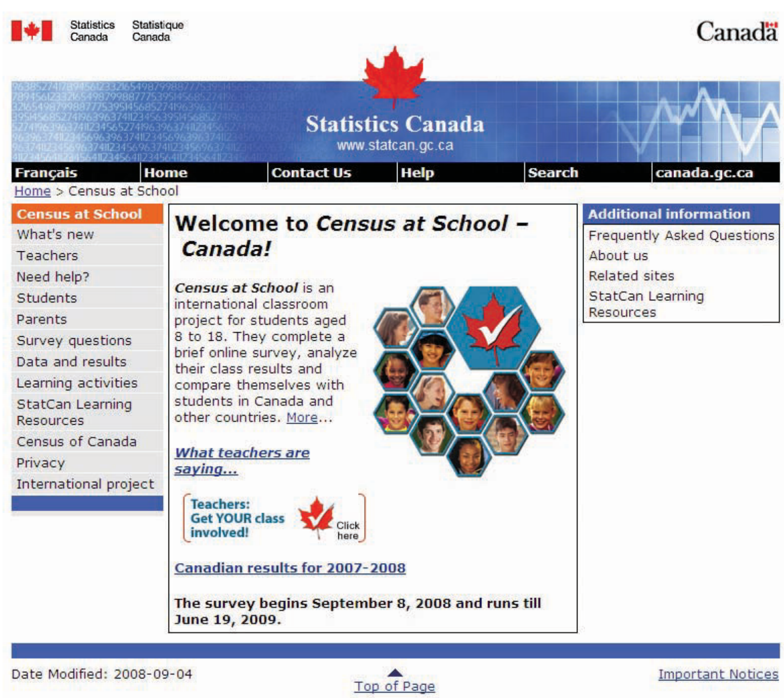
What is your favourite subject?
How much pressure do you feel because of schoolwork?
What is your favourite physical activity?

There is a link to data from other participating countries such as South Africa and New Zealand.

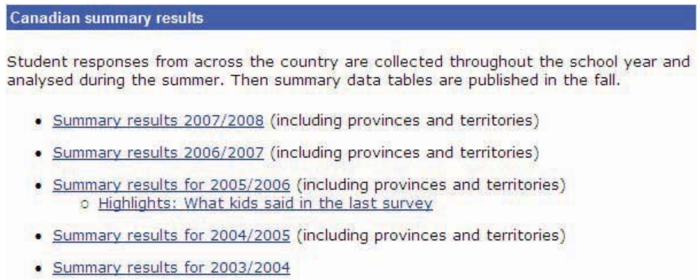
To use *Census at School*, follow these steps:

1. Open the website. Your teacher will give you the address. Select the appropriate language.

2. You will see the *Census at School* homepage. Click on *Data and results* located in the table on the left of the screen.



3. Under *Canadian summary results*, you can access data collected over the past several years. Select the latest summary results.

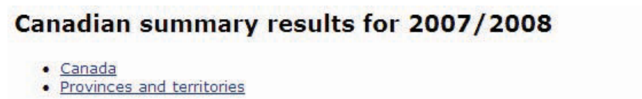


Canadian summary results

Student responses from across the country are collected throughout the school year and analysed during the summer. Then summary data tables are published in the fall.

- Summary results 2007/2008 (including provinces and territories)
- Summary results 2006/2007 (including provinces and territories)
- Summary results for 2005/2006 (including provinces and territories)
  - Highlights: What kids said in the last survey
- Summary results for 2004/2005 (including provinces and territories)
- Summary results for 2003/2004

Note: These results can be viewed by provinces and territories by selecting the *Provinces and territories* link at the top of the page.



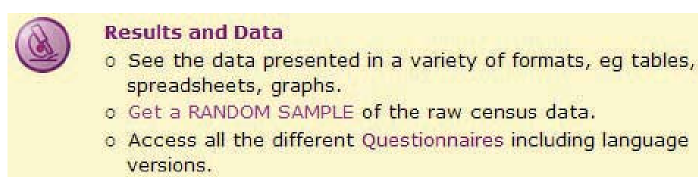
Canadian summary results for 2007/2008

- Canada
- Provinces and territories

4. Use the site to answer these questions.
   a) What is the most popular mode of transportation to school for Canadian youth? Is this the most popular mode of transportation for each province or territory? Explain.
   b) What are the two most popular methods of communication for Canadian students?
   c) Which type of charity would most students support if they had $1000 to donate?
   d) What percent of students have been bullied 10 times or more in the last month in Manitoba?

5. To find data from other countries, click on the *International project* link on the left of the screen. Next, click on the *CensusAtSchool* link. Then on the next screen click on the link provided.

6. Select the United Kingdom (UK). Click on the link to *Results and Data*. Then, select the *Phase 7 Results* link. What do 14-year-olds feel is the most important issue facing the UK today?



Results and Data

- See the data presented in a variety of formats, eg tables, spreadsheets, graphs.
- Get a RANDOM SAMPLE of the raw census data.
- Access all the different Questionnaires including language versions.

Check

Return to the data for Canadian youth.

1. Select a topic that interests you, and report on your findings.

2. How many elementary and secondary students have participated in Canada's *Census at School*? Do you think this sample would produce valid conclusions? Explain.

Mid-Unit Review

9.1

1. Before a security company hires someone, that person must pass a lie-detector test. Suppose that a lie detector has a 0.9 probability of identifying a lie.
A person being tested thinks that if he lies 10 times, 9 of those lies will be detected.
a) Name one assumption the person is making.
b) Explain how the predicted outcome might change if the assumption changes.

2. Due to global warming, the West Antarctic Ice Sheet (WAIS) could melt and raise sea levels. Some scientists think there is a 1 in 20 chance that WAIS will collapse in the next 200 years. Explain how this statistic could be used to support opposing positions about the effects of global warming.

9.2

3. Ca Bol surveys a group of people to find out how they feel about students listening to music while studying.
a) Write a question Ca Bol could use to influence:
  i) the responses in favour of students listening to music while studying
  ii) the responses to oppose students listening to music while studying
b) Write a question that does not show a bias. Explain how this question is more suitable than the questions in part a.

4. Suppose your teacher conducts a survey in class about student smoking.
a) What problems might arise?
b) How would these problems affect the data collected?

5. Ahmed wanted to find out if a person's years of post-secondary education is related to how much the person earns.
a) Describe problems Ahmed might have to overcome related to:
  i) privacy
  ii) cultural sensitivity
  iii) use of language
  iv) cost and time
b) Describe the effect each problem may have on Ahmed's results.

6. Describe a situation where the timing of a question may influence the responses.

9.3

7. Which students in your school would you survey for their opinions on each topic?
a) the quality of cafeteria service
b) the cost of a gym uniform
c) the number of student parking spaces
d) the school spirit at football games

8. For each situation, explain why data were not collected by a census.
a) the number of Canadian families with internet access
b) the average cost of DVD players
c) the average mass of a Northern pike in Misaw Lake, Saskatchewan

9. For each topic, would you collect data using a census or a sample?
Justify your choice.
a) to determine the average height of a grade 9 student in your class
b) to determine the reaction to new traffic laws in your province or territory

9.4 Selecting a Sample

FOCUS

- Understand and choose an appropriate sample.

When we cannot survey an entire population, we choose a sample from the population.
When a political party wants to determine if its candidate is likely to win the next territorial election, it may conduct a telephone survey of a sample of voters. How could the party ensure that the sample is representative of the population?

Investigate

2

Suppose a school considers making the cafeteria food more healthy. The school would like you to determine the reactions of the school population.

List 3 ways you could select the sample of people to be questioned.
Discuss and record the advantages and disadvantages of each way.
Choose the most appropriate way of selecting your sample.

Reflect & Share

Share your 3 ways of selecting a sample with another pair of students.
Did you come up with any of the same ways?
Which ways had similar advantages and disadvantages?
Discuss and select the best way from all the possibilities.
Justify your choice.

Connect

Here are some common sampling methods:

Simple random sampling
Each member of the population has an equal chance of being selected.
For example, to select a random sample of 5 students from your math class, each student is assigned a number and 5 numbers are drawn from a hat.

Systematic or interval sampling
Every *n*th member of the population is selected.
This method is often used in manufacturing; for example, every 20th product in an assembly line is tested for quality. If the item is destroyed or unusable after being sampled, then the sample is a *destructive sample.*

Cluster sampling
Every member of each randomly chosen group of the population is selected.
For example, each grade represents a group of the school population. One grade in your school is chosen randomly, and all students in that grade are selected.

Self-selected sampling
Only members who are interested and volunteer will participate.
For example, if a radio station conducts a telephone survey, only people who are interested will call.

Convenience sampling
Only members of the population who are convenient to include are selected.
For example, for a survey about grocery shopping habits, people in a grocery store are approached and questioned.

Stratified random sampling
Some members from each group of the population are randomly selected.
For example, 5 randomly chosen students from each grade in a school could be selected, even if each grade has a different number of students.

Example 1 Identifying Appropriate Samples

The student leadership team wants to find out if students would like the cafeteria to have longer hours. Several sampling methods were suggested. Explain whether each sample is appropriate.

a) Every student's name is put into a box, and 100 names are selected randomly to be surveyed.
b) Every 5th person entering the school is selected.
c) Each person on the leadership team asks her or his friends.
d) An announcement is made asking anyone who wishes to participate to fill in a ballot.

A Solution

	Sampling Method	Is the Sample Appropriate?
a)	Simple random sampling	The sample is appropriate because every student has an equal chance of being selected.
b)	Systematic sampling	The sample may or may not be appropriate depending on when you ask students. If you ask students who arrive early in the morning, then these students may appreciate the cafeteria having longer hours. The opinions of these students would likely not be representative of the entire student population.
c)	Convenience sampling	The sample is likely not appropriate because friends often have similar views on issues.
d)	Self-selected sampling	The sample is likely not appropriate because only students who have strong opinions about this topic may respond.

Example 2 Choosing Appropriate Samples

A company packages boxes of granola bars. The quality-control manager inspects the first 5 boxes each morning to ensure that each has the same number and types of granola bars.

a) Is this a good way of ensuring quality control? Explain.
b) Suggest 2 other methods of sampling that would be appropriate. Explain why each is appropriate.

A Solution

a) This may not be a good way of ensuring quality control because the people working on the assembly line may be more alert in the morning. So, the boxes filled in the mornings may pass inspection. However, the boxes made later in the day, which may not meet the manager's standards, are never inspected.

b) Systematic sampling would allow the manager to inspect several boxes throughout the day. For example, each 50th box could be inspected. Simple random sampling throughout the day would also be appropriate because it ensures each box has an equal chance of being selected.

Discuss the ideas

1. Which sampling methods are least likely to produce valid conclusions? Explain.

2. Which sampling methods are most likely to produce valid conclusions? Explain.

Practice

Check

3. Identify a potential problem with each sampling method.
   a) Suppose you want to know whether most people enjoy shopping. You survey the shoppers at a local mall.
   b) The cook in the school cafeteria surveys the teachers to find out which items to sell.
   c) To determine public opinion on the effectiveness of the local police force, residents in the area with the greatest crime rate are surveyed.
   d) To find out about the exercise habits of Canadian teenagers, a fitness magazine asks its readers to email information about their exercise habits.

Apply

4. Explain whether each sample is appropriate. Justify your answer.
   a) A TV show asks viewers to text their opinions about the decreased speed limit in town.
   b) To determine if customers are pleased with the service in a restaurant, every 8th customer is polled on a given day.
   c) Fifty student ID numbers were randomly selected by a computer. The students with these ID numbers were surveyed about a new school policy.
   d) Ten students were randomly selected from each grade to estimate how many students in the school cycle to school.
   e) To determine if all physical education students would prefer to go skiing or skating on a field trip, one gym class was randomly selected from a list and each student in the class was polled.

5. a) In each case, will the selected sample represent the population? Explain.
      i) To find out if the arena should offer more public skating times, a survey is posted on a bulletin board in the arena and left for patrons to complete.
      ii) To find out the favourite breakfast food of grade 9 students, a survey of 300 randomly-selected grade 9 students was conducted.
      iii) To find out if the soccer league should buy new uniforms for the players, 20 parents of the students in the soccer league were surveyed.
   b) If the sample does not represent the population, suggest another sample that would. Describe how you would select that sample.

6. Describe an appropriate sampling method for each situation. Justify your answers.
   a) The Prime Minister wants to know citizens' opinions about the new budget.
   b) The school newspaper wants to poll students to predict who will be elected student president.

7. To determine citizens' view of new parking fines, the mayor invites listeners to call in during a radio show. Do you think the results will accurately reflect the opinions of all citizens? Explain.

8. **Assessment Focus** Suppose you want to find out how people feel about lowering the age at which teens can drive.
   a) Describe a sampling method that would not lead to valid conclusions. Justify your choice.
   b) Describe a sampling method you might use, and justify your choice.

9. For each topic, identify a sample of people whose opinions would bias the survey results. Explain your choice.
   a) whether fur from animals should be used for coats
   b) whether households should be fined for not recycling

10. A survey reports: Fifty Canadians say that the most important issue Canadians face is global warming.
   a) Do you think this sample is representative of the population? Explain.
   b) How might this sample have been selected?
   c) Suppose you were to repeat the survey. How would you select a sample? Explain how your choice of sample would represent the population.

Take It Further

11. Some sampling methods produce invalid conclusions more often than they produce valid conclusions. Which sampling methods do you think fit this description? Why do you think these sampling methods are still used?

12. a) Explain how you might obtain each sample.
      i) a simple random sample from the school population
      ii) a systematic sample of cell phones from a factory
      iii) a cluster sample of teenagers from your town
      iv) a stratified random sample of apple trees in an orchard
   b) Suggest a topic of data collection for each sample in part a.

Reflect

When you select a sample to represent a population, what factors must you consider?

Using Spreadsheets and Graphs to Display Data

FOCUS

- Display data on graphs using spreadsheets.

You can use a graph to display your data in a way that is clear and easy to understand.

Spreadsheet software can be used to record and graph data.

These data come from the *Census at School* website:

Which method do you use most often to communicate with friends?

Method of communication	Girls	Boys	All students
		%	
Internet chat or MSN	36.11	35.26	35.7
In person	30.02	35.51	32.65
Telephone (land line)	15.61	13.5	14.6
Cell phone	8.91	7.66	8.31
Text messaging	6.59	3.77	5.23
E-mail	1.73	2.20	1.96
Other	1.03	2.11	1.55

Notes: Secondary students only.
Methods of communication appear in order of frequency for all students.

Source: Statistics Canada, Census at School, 2006/2007.

1. Enter the *Method of communication* and the percent of *All students* into columns and rows.
2. Highlight the data including the column heads. Click the graph/chart icon. Select the circle graph, which is sometimes called a pie chart. Label the graph and all sectors of the circle.
   Your graph might look like this:

Method of communication	All students
Internet chat or MSN	35.7
In person	32.65
Telephone (land line)	14.6
Cell phone	8.31
Text messaging	5.23
E-mail	1.96
Other	1.55